Sub-Finsler minimal and constant mean curvature surfaces in the first Heisenberg group

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Anisotropic isoperimetric problems
and related topics

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Introduction

Anisotropic functionals in $\mathbb{R}^n$

Given an (asymmetric) norm $||\cdot||_K$ whose unit ball is a convex body $K \subset \mathbb{R}^n$, the area of an oriented Lipschitz boundary S is given by

$$A_K(S) = \int_S ||N||_{K,*} dS, \qquad (*)$$

where N is the outer unit normal to S, $||u||_{K,*} = \sup_{v \in K} \langle u, v \rangle$ is the dual norm of K, and dS is the standard area element on S.

(*) is an anisotropic energy which models the shape of a crystal minimizing Gibbs' free energy (1875). The minimizers of the problem when K is a polyhedron were described by Wulff (1895).

Introduction

Anisotropic functionals in $\mathbb{R}^n$

For $S = \partial E$ regular enough, $A_K(S)$ coincides with the Minkowski content

$$M(E,K) = \liminf_{r\to 0} \frac{|E+rK| - |E|}{r}.$$

Brunn-Minkowski inequality implies

$$|E+rK|^{1/n} - |E|^{1/n} \geq r|K|^{1/n}.$$

Dividing by r and taking limits when $r \to 0$, since $M(K,K) = n|K|$,

$$\frac{M(E,K)}{|E|^{(n-1)/n}} \geq \frac{M(K,K)}{|K|^{(n-1)/n}}.$$

So K minimizes the functional (*) for given volume. The set K is known as the *Wulff shape* of (*)

Introduction

Wulff shapes in $\mathbb{R}^n$

- The functional (*) is used in crystallography (Gibbs free energy). Wulff gave a construction to obtain K from the dual norm $||\cdot||_{K,*}$
- Use of Brunn-Minkowski to obtain a solution by Dinghas (1944)
- Mathematical problem considered by Taylor (1978), Fonseca (1991) and Fonseca-Müller (1991)

Introduction

Wulff shapes in $\mathbb{R}^n$

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Goal

Explore a similar anisotropic functional in the Heisenberg group $\mathbb{H}^1$. There is no useful Brunn-Minkowski inequality (Leonardi-Masnou, 2005)

The sub-Finsler perimeter

The Heisenberg group $\mathbb{H}^1$

$\mathbb{H}^1 = (\mathbb{R}^3, *)$, where $*$ is the product

$$(x, y, t) * (x', y', t') := (x + x', y + x', t + t' + (x'y - xy')),$$

A basis of left invariant vector fields is given by

$$X = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

X, Y generate a non-integrable horizontal distribution $\mathcal{H}$ ($[X, Y] = -2T$), $\langle \cdot, \cdot \rangle$ is the Riemannian metric so that X, Y, T is orthonormal basis

Sub-Finsler norms in $\mathbb{H}^1$

A planar norm $|| \cdot ||_K$ ($K \subset \mathbb{R}^2$) is extended to a left-invariant norm in $\mathcal{H}$

$$(||fX + gY||_K)_p = ||(f(p), g(p))||_K.$$

The sub-Finsler perimeter

Sub-Finsler K-perimeter in $\mathbb{H}^1$

Let $E \subset \mathbb{H}^1$ be a measurable set, $||\cdot||_K$ the left-invariant norm associated to $K \subset \mathbb{R}^2$, and $\Omega \subset \mathbb{H}^1$ an open subset. We say that E has locally finite K-perimeter in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$|\partial E|_K(V) = \sup\left\{ \int_E \operatorname{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), ||U||_{K,\infty} \leq 1 \right\} < +\infty.$$

div is the Riemannian divergence and $d\mathbb{H}^1$ the Haar (Riemnnian or Lebesgue) measure of $\mathbb{H}^1$, $\mathcal{H}_0^1(V)$ the set of horizontal fields with compact support in V

Remark

If K is the closed unit disc D centered at 0, $|\partial E|_D$ is the classical sub-Riemannian perimeter of E.

The sub-Finsler perimeter

Sub-Finsler area in $\mathbb{H}^1$

If $S = \partial E$ is a Euclidean Lipschitz hypersurface then

$$|\partial E|_K(V) = \int_{S \cap V} ||N_h||_{K,*} \, dS$$

where N_h is the horizontal projection of the outer unit normal N and dS is the Riemannian measure of S (computed with the Riemannian metric g so that X, Y, T is orthonormal)

The sub-Finsler perimeter

Problems

- Critical points of K-perimeter (variation formulas)
- Is there a mean curvature function?
- Geometric conditions on critical points
- Existence of minimizers to variational problems involving the K-perimeter
- Regularity of such minimizers

The sub-Finsler perimeter

Recent works

- A.P. Sánchez, Ph.D. Thesis, Tufts U., 2017
- J. Pozuelo, M. Ritoré, Pansu-Wulff shapes in $\mathbb{H}^1$. Adv. Calc. Var. (to appear), arXiv:2007.04683
- V. Franceschi, R. Monti, A. Righini, and M. Sigalotti, The isoperimetric problem for regular and crystalline norms in $\mathbb{H}^1$, arXiv:2007.11384
- G. Giovannardi, M. Ritoré, Regularity of Lipschitz boundaries with prescribed sub-Finsler mean curvature in the Heisenberg group. J. Differential Equations, 2021, arXiv:2010.14882
- J, Pozuelo, Existence of isoperimetric regions in sub-Finsler nilpotent groups, arXiv:2103.06630

The sub-Finsler perimeter

Recent works

- G. Giovannardi, M. Ritoré, The Bernstein problem for Euclidean Lipschitz surfaces in the sub-Finsler Heisenberg group, arXiv:2105.02179
- G. Giovannardi, J. Pozuelo, M. Ritoré, Area-minimizing horizontal graphs with low-regularity in the sub-Finsler Heisenberg group $\mathbb{H}^1$, arXiv:2204.03474.
- G. Giovannardi, A. Pinamonti, J. Pozuelo, S. Verzellesi, The prescribed mean curvature equation for t-graphs in the sub-Finsler Heisenberg group $\mathbb{H}^n$, arXiv:2207.13414.

First variation of perimeter for C^2 boundaries

The singular set

Given a C^1 surface $S \subset \mathbb{H}^1$, its singular set $S_0 \subset S$ is the set of points where T_pS coincides with the horizontal distribution ($N_h = 0$).

$$A_K(S_0) = \int_{S_0} ||N_h||_{K,*} dS = 0$$

The regular set $S \setminus S_0$ is foliated by horizontal curves with tangent vector Z

The class C^2_+

We say that a convex body is of class C^2_+ if the boundary of K is C^2 and has positive curvature everywhere.

If K is of class C^2_+, for every $u \neq 0$ there is a unique $\pi_K(u)$ such that $\langle \pi_K(u), u \rangle = ||u||_{K,*}$. It is geometrically clear that $\pi_K = N_K^{-1}$, the inverse of the Gauss map of N_K.

Theorem (Pozuelo-Ritoré, 2020)

Let K be of class C^2_+ and S be a C^2 surface in $\mathbb{H}^1$. Let U be a C^2 vector field with compact support on $S \setminus S_0$, and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated flow. Let $\eta = \pi_K(N_h)$. Then

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S \langle U, N\rangle \langle D_Z\eta, Z\rangle \, dS.$$

K-mean curvature H_K

We let $H_K = \langle D_Z\eta, Z\rangle$. This is an ODE along horizontal curves in $S \setminus S_0$

Theorem (Pozuelo-Ritoré, 2020)

Let S be a C^2 surface without singular points and constant mean curvature H_K.

- If $H_K > 0$ then $S \setminus S_0$ is foliated by horizontal liftings of translations of the circle $||\cdot||_K = 1/H_k$.
- If $H_K = 0$ then $S \setminus S_0$ is foliated by horizontal straight lines.

First variation of perimeter for C^2 boundaries

Is H_K constant a sufficient condition for a critical point?

Unfortunately not. There is an additional condition involving the singular set S_0

- By Franceschi et al. (arXiv:2007.11384) the singular set S_0 of a C^2 surface with CMC is composed of isolated points and singular curves
- By Giovannardi et al. (arXiv:2204.03474) the horizontal curves in $S \setminus S_0$ must meet the singular curves at given angles depending on K and the directions (condition obtained in the sub-Riemannian case by Cheng-Hwang-Yang (2007))

The Pansu-Wulff shapes

Definition

Let K of class C^2_+ and parametrize ∂K by an L-periodic curve $\gamma : \mathbb{R} \to \mathbb{R}^2$. For any $u \in \mathbb{R}$ consider the horizontal lifting $\Gamma_{\gamma(u)} : \mathbb{R} \to \mathbb{H}^1$ of the curve $t_{-\gamma(u)}(\gamma)$ with initial point $(0, 0, 0)$. Define

$$\mathbb{B}_K = \bigcup_{u \in [0,L)} \Gamma_{\gamma(u)}([u, u + L]).$$

We shall refer to $\mathbb{B}_K$ as the *Pansu-Wulff shape* associated to the left-invariant norm $|| \cdot ||_K$. Its boundary $\mathbb{S}_K = \partial \mathbb{B}_K$ will be called the *Pansu-Wulff sphere*.

In the sub-Riemannian case, the corresponding sphere is known as Pansu sphere.

The Pansu-Wulff shapes (regularity properties)

Parameterization of the Wulff shape

Given any convex body $K \subset \mathbb{R}^2$ with $0 \in \operatorname{int}(K)$ we parameterize ∂K as

$$\gamma(s) = \big(x(s), y(s)\big) = r(s)\,\big(\sin(s), \cos(s)\big), \quad s \in \mathbb{R}.$$

where $r(s) = \rho(\sin(s), \cos(s))$ and ρ is the radial function of K.

Then we have the following parameterization of $\mathbb{S}_K$.

$$\begin{aligned} x(u,v) &= r(u+v)\sin(u+v) - r(v)\sin(v), \\ y(u,v) &= r(u+v)\cos(u+v) - r(v)\cos(v), \\ t(u,v) &= r(v)r(u+v)\big(\sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big) \\ &\qquad + \int_v^{u+v} r^2(\xi)\,d\xi. \end{aligned}$$

Regularity properties follow from this expression. Also convergence in Hausdorff distance of Wulff shapes

The Pansu-Wulff shapes

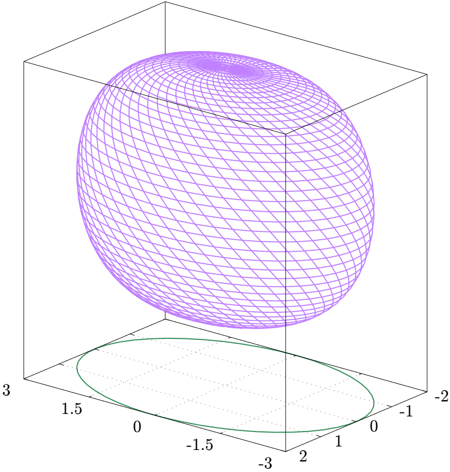


Figure: The Wulff shape associated to the norm $||\cdot||_a = ((x_1/a_1)^2 + (x_2/a_2)^2)^{1/2}$ with $a = (1, 1.5)$. Observe that the projection to the horizontal plane $t = 0$ is an ellipse with semiaxes of lengths 2 and 3.

The Pansu-Wulff shapes

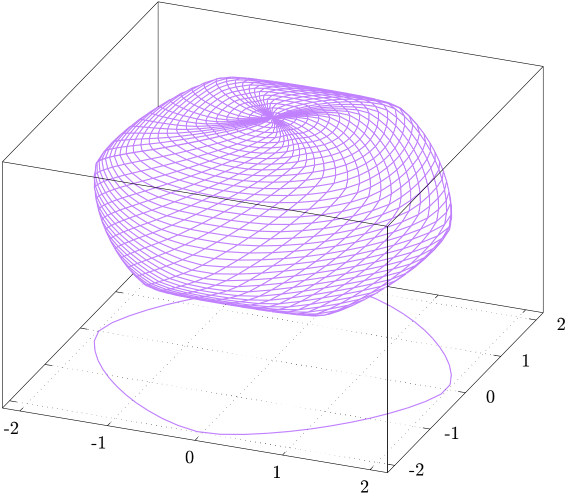


Figure: The Wulff shape $\mathbb{S}_{K_r}$ for the r-norm, $r = 1.5$. The horizontal curve is the projection of the equator to the plane $t = 0$. Since the r-norm is symmetric, the Wulff shape projects to the set $||\cdot||_r \leq 2$ in the $t = 0$ plane.

The Pansu-Wulff shapes

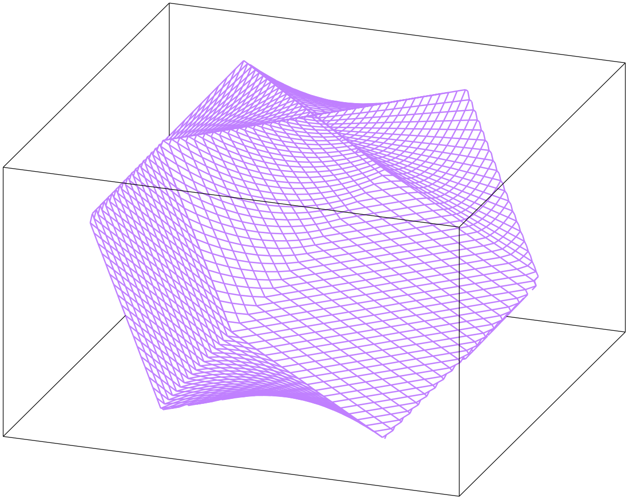


Figure: The ball $\mathbb{B}_1$ obtained as Hausdorff limit of the Wulff shapes $\mathbb{B}_{K_r}$ of the r-norm when r converges to 1

The Pansu-Wulff shapes

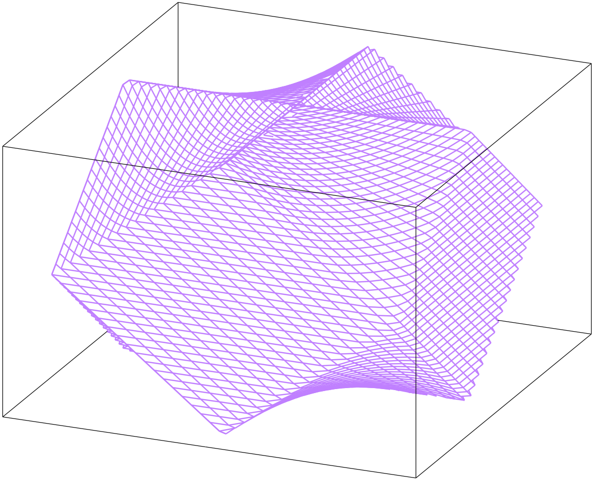


Figure: The ball $\mathbb{B}_\infty$ obtained as Hausdorff limit of the Wulff shapes $\mathbb{B}_{K_r}$ of the r-norm when r converges to ∞

The Pansu-Wulff shapes

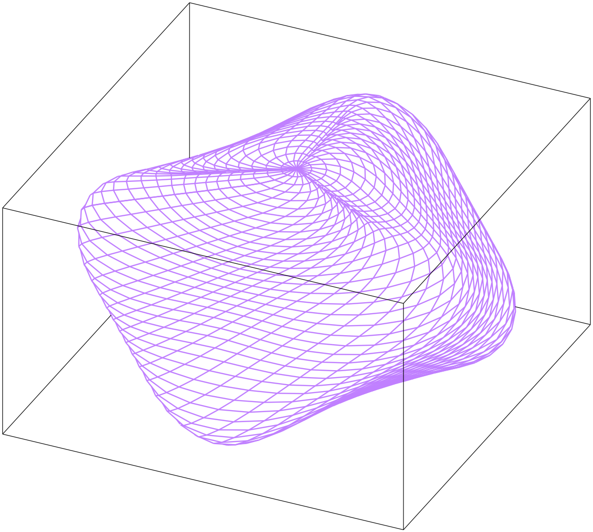


Figure: The Wulff shape $\mathbb{B}_{T,r}$ for the norm $||\cdot||_{T,r}$, with $r = 2$.

The Pansu-Wulff shapes

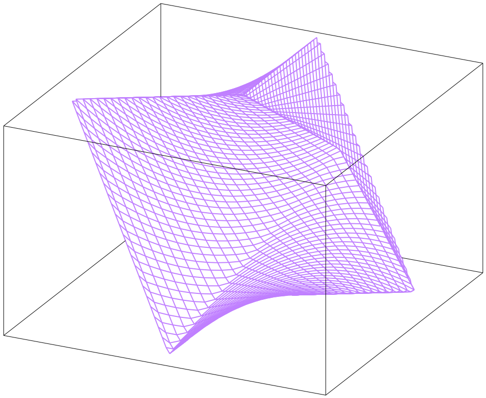


Figure: The ball $\mathbb{B}_T$ obtained as limit of the Wulff shapes $\mathbb{B}_{T,r}$ when $r \to \infty$.

Pansu-Wulff shapes: minimization properties

(Franceschi et al., arXiv:2007.11384)

Pansu-Wulff boundaries are the only C^2 stationary points of area under a volume constraint. Proof after Ritoré-Rosales (2005)

Theorem (Pozuelo-Ritoré, arXiv:2007.04683)

Let $||\cdot||_K$ be the norm associated to an strictly convex body $K \subset \mathbb{R}^2$ with C^2 boundary. Let $r > 0$ and $h : rK_0 \to \mathbb{R}$ a C^0 function. Consider a subset $E \subset \mathbb{H}^1$ with finite volume and K-perimeter such that

$$\text{graph}(h) \subseteq E \subset rK_0 \times \mathbb{R}.$$

Then

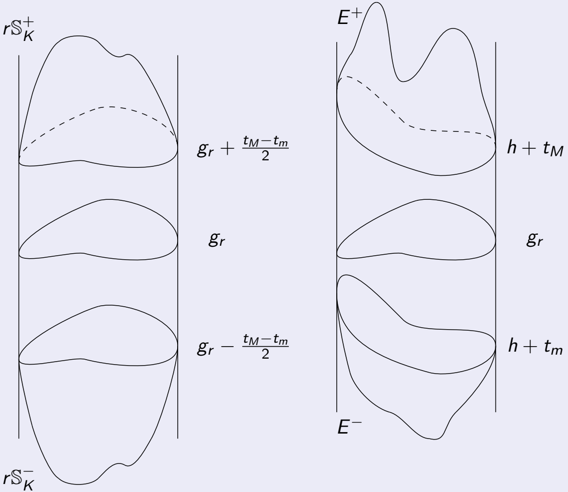
$$|\partial E|_K \geq |\partial \mathbb{B}_E|_K,$$

where $\mathbb{B}_E$ is the Wulff shape with the same volume as E.

Pansu-Wulff shapes: minimization properties

Proof

We apply a calibration argument



Regularity properties

Sets with prescribed mean curvature

Given $f \in C(\Omega)$, $E \subset \Omega$ has prescribed K-mean curvature f in $\Omega \Leftrightarrow E$ is a critical point of the functional

$$B \mapsto |\partial E|_K(B) - \int_{E \cap B} f \, d\mathbb{H}^1, \quad \forall\, B \subset \Omega \text{ bounded open set} \tag{1}$$

If $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface then (1) $\Leftrightarrow E$ is a critical point of the functional

$$B \mapsto A_K(S \cap B) - \int_{E \cap B} f \, d\mathbb{H}^1, \quad \forall\, B \subset \Omega \text{ bounded open set}$$

Remarks

- Perimeter-minimizing sets have prescribed K-mean curvature 0
- Isoperimetric boundaries have prescribed constant K-mean curvature

Regularity of the non-singular set (Giovannardi-Ritoré, 2021)

Let K be a convex body of class C^2_+, $\Omega \subset \mathbb{H}^1$ an open set and $E \subset \Omega$ a set of prescribed K- mean curvature $f \in C^0(\Omega)$ with Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then the horizontal curves Γ of $S \cap \Omega$ are of class C^2.

$\mathbb{H}$-regularity of the boundary means that the surface is locally the level set of a continuous function with non-vanishing continuous horizontal gradient.

Proof

Quite technical and follows from localizing the first variation along horizontal curves of the surface

The Bernstein problem

Bernstein's Theorem (Giovannardi-Ritoré, 2022)

Let $S \subset \mathbb{H}^1$ be a complete, stable, Euclidean Lipschitz and $\mathbb{H}$-regular surface without singular points. Then S is a vertical plane.

The proof is based on the second variation of the K-perimeter

$$\int_S \left(Z(f)^2 + 4 \left(Z \left(\frac{\langle N, T \rangle}{|N_h|} \right) - \frac{\langle N, T \rangle^2}{|N_h|^2} \right) f^2 \right) \frac{|N_h|}{\kappa(\pi(\nu_h))} dS.$$

We localize the second variation along a horizontal line (where $\kappa(\pi(N_h))$ is constant because $\pi(N_h)$ is constant)