# Pansu-Wulff shapes in the Heisenberg group

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# Introduction

## The isoperimetric inequality in $\mathbb{R}^n$

The isoperimetric inequality in $\mathbb{R}^n$ reads

$$\text{boundary measure of } E \geqslant C(n)\,|E|^{(n-1)/n},$$

where $|E|$ is the volume of a measurable set $E$. The constant $C(n)$ depends on the dimension and it is equal to $P(B)/|B|^{(n-1)/n}$, where $B$ is any ball in $\mathbb{R}^n$.

It can be also proven that equality holds if and only if $E$ "is a ball".

## What is the boundary measure of $E$?

When the boundary of $E$ is of class $C^2$ there are three equivalent defs:

1. Its area
2. The Minkowski content

$$M(E) = \lim_{r \to 0} \frac{|E + r\overline{B}(0,1)| - |E|}{r},$$

where, for $r > 0$, $E + r\overline{B}(0,1)$ is the tubular neighborhood of $E$ of radius $r$, defined by $E_r = \{p \in \mathbb{R}^n : d(p,E) \leqslant r\}$.

3. The perimeter of $E$ in the sense of Cacciopoli and De Giorgi:

$$P(E) = \sup \left\{ \int_E \operatorname{div} X : X \in \mathfrak{X}_0^1(\mathbb{R}^n), ||X|| \leqslant 1 \right\},$$

where $\mathfrak{X}_0^1(\mathbb{R}^n)$ is the set of $C^1$ vector fields on $\mathbb{R}^n$ with compact support.

## Observation

When $\partial$ is of class $C^2$ 1 and 2 are equivalent by Weyl's tube formula, and 1 and 3 as an application of the divergence theorem.

For the relation between perimeter in the sense of Cacciopoli and De Giorgi and Minkowski content (for general sets) one should take a look at

1. L. Ambrosio et al., *Perimeter as relaxed Minkowski content in metric measure spaces*, Nonlinear Anal. 153 (2017), 78–88.
2. M. Ritoré, *Isoperimetric inequalities in Riemannian manifolds*, Progress in Mathematics 348, Birkhauser Verlag, 2023

## Proof of the isoperimetric inequality using the Brunn-Minkowski inequality

Denote $\overline{B}(0,1)$ by $B$. The Brunn-Minkowski inequality implies

$$|E + rB|^{1/n} - |E|^{1/n} \geqslant r|B|^{1/n}.$$

Dividing by $r$ and taking limits when $r \to 0$, since $M(B) = n|B|$,

$$\frac{M(E)}{|E|^{(n-1)/n}} \geqslant \frac{M(B)}{|B|^{(n-1)/n}} = C(n).$$

# Anisotropic functionals

## Anisotropic functionals in the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$

Given an (asymmetric) norm $|| \cdot ||_K$ whose unit ball is a convex body $K \subset \mathbb{R}^n$, the *area* $A_K$ of an oriented Lipschitz boundary $S$ is defined by

$$A_K(S) = \int_S ||N||_{K,*} dS, \qquad (*)$$

where $N$ is the outer unit normal to $S$, $||u||_{K,*} = \sup_{v \in K} \langle u, v \rangle$ is the dual norm of $K$, and $dS$ is the standard area element on $S$.

When $K$ is the unit ball, $||N||_{K,*} \equiv 1$, and $A_K$ is the standard area.

A *crystal* is a minimizer of (*) with fixed volume. Wulff (1895) first gave a construction to obtain $K$ from $|| \cdot ||_{K,*}$.

This is a different notion of boundary area that leads to other isoperimetric inequalities.

## Equivalences

For a bounded set $E \subset \mathbb{R}^n$ with $C^2$ boundary $S$ the following quantities coincide.

1. The area $A_K(S)$
2. The Minkowski content

$$M(E,K) = \lim_{r\to 0} \frac{|E + rK| - |E|}{r}.$$

3. The $K$-perimeter

$$P_K(E) = \sup\Big\{ \int_E \operatorname{div} X \,:\, X \in \mathfrak{X}_0^1(\mathbb{R}^n), ||X||_K \leqslant 1 \Big\},$$

where $\mathfrak{X}_0^1$ is the set of $C^1$ vector fields on $\mathbb{R}^n$ with compact support.

# An anisotropic isoperimetric inequality obtained from the Brunn-Minkowski inequality

## Proof

Brunn-Minkowski inequality implies

$$|E+rK|^{1/n}-|E|^{1/n}\geqslant r|K|^{1/n}.$$

Dividing by $r$ and taking limits when $r\to 0$, since $M(K,K)=n|K|$,

$$\frac{M(E,K)}{|E|^{(n-1)/n}}\geqslant\frac{M(K,K)}{|K|^{(n-1)/n}}=C_K.$$

So $K$ minimizes the functional (*) for volume $|K|$. The set $K$ is known as the *Wulff shape* of (*)

## Wulff shapes in $\mathbb{R}^n$

- The functional (*) defiing $A_K$ is used in crystallography (Gibbs free energy) to model the equilibrium shape of a crystal of fixed volume inside a separate phase. Wulff gave a construction to obtain $K$ from the dual norm $||\cdot||_{K,*}$
- Dinghas (1944) was the first to use Brunn-Minkowski inequality to obtain a solution of this problem.
- Mathematical problem further considered by Taylor (1978), Fonseca (1991) and Fonseca-Müller (1991), using tools of Geometric Measure Theory.

# The Heisenberg group $\mathbb{H}^1$

## The Heisenberg group $\mathbb{H}^1$

$\mathbb{H}^1$ is the Lie group $(\mathbb{R}^3, *)$, where $*$ is the non-abelian product

$$(x, y, t) * (x', y', t') := (x + x', y + x', t + t' + (x'y - xy')),$$

A frame of left-invariant vector fields is given by

$$X = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

$X, Y$ generate a non-integrable distribution $\mathcal{H}$ (the *horizontal* distribution).

The Haar measure is the Lebesgue measure.

The non-homogeneous dilations $h_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$ preserve the horizontal distribution.

## Left-invariant Finsler norms in the horizontal distribution $\mathbb{H}^1$

Identifying $\mathbb{R}^2$ with $\mathcal{H}_0$ it is enough to consider a planar norm $||\cdot||_K$ associated to a convex body $K \subset \mathbb{R}^2$. This norm can be extended to a left-invariant one in $\mathcal{H}$ by means of the equality

$$(||fX + gY||_K)_p = ||(f(p), g(p))||_K.$$

## Sub-Finsler $K$-perimeter in $\mathbb{H}^1$

Let $E \subset \mathbb{H}^1$ be measurable, $||\cdot||_K$ the left-invariant norm associated to $K \subset \mathbb{R}^2$. Define the perimeter $P_K$ of $E$ as:

$$P_K(E) = \sup\left\{ \int_E \operatorname{div}(U)\, d\mathbb{H}^1 \,:\, U \in \mathcal{H}_0^1(\mathbb{H}^1), ||U||_K \leqslant 1 \right\}.$$

Here div is the Riemannian divergence ($X$, $Y$, $T$ orthonormal frame), $d\mathbb{H}^1$ the Haar measure, and $\mathcal{H}_0^1(\mathbb{H}^1)$ the set of horizontal vector fields with compact support in $\mathbb{H}^1$.

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## Some properties

1. $P_K(h_\lambda(E)) = \lambda^3 P_K(E)$,
2. $|h_\lambda(E)| = \lambda^4 |E|$.

This implies that the optimal isoperimetric inequality in $(\mathbb{H}^1, ||\cdot||_K)$ must be of the form $P_K(E) \geqslant C\,|E|^{3/4}$.

## Sub-Finsler area in $\mathbb{H}^1$

If $S = \partial E$ is Lipschitz then the divergence theorem implies

$$P_K(E) = \int_S ||N_h||_{K,*}\, dS$$

where $N_h$ is the horizontal projection of the outer unit normal $N$ and $dS$ is the Riemannian measure of $S$ (computed with the Riemannian metric g so that $X, Y, T$ is orthonormal)

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## Minkowski content and Brunn-Minkowski

There is a notion of Minkowski content. Unfortunately there is no useful Brunn-Minkowski inequality since the exponent is not the (Leonardi and Masnou (2005)).

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## Problem

How do we identify candidates to be minimizers of $A_K$ under a volume constraint?

# First variation of $P_K$ for $C^2$ boundaries

## The singular set

Given a $C^1$ surface $S \subset \mathbb{H}^1$, its singular set $S_0 \subset S$ is the set of points where $T_pS$ coincides with the horizontal distribution ($N_h = 0$).

$$A_K(S_0) = \int_{S_0} ||N_h||_{K,*} dS = 0$$

The regular set $S \setminus S_0$ is foliated by horizontal curves with tangent vector $Z$

## The class $C^2_+$

We say that a convex body is of class $C^2_+$ if the boundary of $K$ is $C^2$ and has positive curvature everywhere.

If $K$ is of class $C^2_+$, for every $u \neq 0$ there is a unique $\pi_K(u)$ such that $\langle \pi_K(u), u \rangle = ||u||_{K,*}$. It is geometrically clear that $\pi_K = N_K^{-1}$, the inverse of the Gauss map of $N_K$.

## Theorem (Pozuelo-Ritoré, 2020)

Let $K$ be of class $C^2_+$ and $S$ be a $C^2$ surface in $\mathbb{H}^1$. Let $U$ be a $C^2$ vector field with compact support on $S \setminus S_0$, and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated flow. Let $\eta = \pi_K(N_h)$. Then

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S \langle U, N\rangle\langle D_Z\eta, Z\rangle\, dS.$$

## $K$-mean curvature $H_K$

We let $H_K = \langle D_Z\eta, Z\rangle$. This is an ODE along horizontal curves in $S \setminus S_0$

## Theorem (Pozuelo-Ritoré, 2020)

Let $S$ be a $C^2$ surface without singular points and constant mean curvature $H_K$.

- If $H_K > 0$ then $S \setminus S_0$ is foliated by horizontal liftings of translations of the circle $||\cdot||_K = 1/H_k$.
- If $H_K = 0$ then $S \setminus S_0$ is foliated by horizontal straight lines.

# First variation of perimeter for $C^2$ boundaries

## Is $H_K$ constant a sufficient condition for a critical point?

Unfortunately not. There is an additional condition involving the singular set $S_0$

- By Franceschi et al. (arXiv:2007.11384) the singular set $S_0$ of a $C^2$ surface with CMC is composed of isolated points and singular curves
- By Giovannardi et al. (arXiv:2204.03474) the horizontal curves in $S \setminus S_0$ must meet the singular curves at given angles depending on $K$ and the directions (condition obtained in the sub-Riemannian case by Cheng-Hwang-Yang (2007))

# The Pansu-Wulff shapes

## Definition

Let $K$ of class $C^2_+$ and parametrize $\partial K$ by an $L$-periodic curve $\gamma : \mathbb{R} \to \mathbb{R}^2$. For any $u \in \mathbb{R}$ consider the horizontal lifting $\Gamma_{\gamma(u)} : \mathbb{R} \to \mathbb{H}^1$ of the curve $t_{-\gamma(u)}(\gamma)$ with initial point $(0,0,0)$. Define

$$\mathbb{B}_K = \bigcup_{u\in[0,L)} \Gamma_{\gamma(u)}([u, u+L]).$$

We shall refer to $\mathbb{B}_K$ as the *Pansu-Wulff shape* associated to the left-invariant norm $||\cdot||_K$. Its boundary $\mathbb{S}_K = \partial\mathbb{B}_K$ will be called the *Pansu-Wulff sphere*.

In the sub-Riemannian case, the corresponding sphere is known as Pansu sphere.

# The Pansu-Wulff shapes (regularity properties)

## Parameterization of the Wulff shape

Given any convex body $K \subset \mathbb{R}^2$ with $0 \in \mathrm{int}(K)$ we parameterize $\partial K$ as

$$\gamma(s) = \big(x(s), y(s)\big) = r(s)\big(\sin(s), \cos(s)\big), \quad s \in \mathbb{R}.$$

where $r(s) = \rho(\sin(s), \cos(s))$ and $\rho$ is the radial function of $K$.

Then we have the following parameterization of $\mathbb{S}_K$.

$$\begin{aligned} x(u,v) &= r(u+v)\sin(u+v) - r(v)\sin(v), \\ y(u,v) &= r(u+v)\cos(u+v) - r(v)\cos(v), \\ t(u,v) &= r(v)r(u+v)\big(\sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big) \\ &\qquad + \int_v^{u+v} r^2(\xi)\,d\xi. \end{aligned}$$

Regularity properties follow from this expression. Also convergence in Hausdorff distance of Wulff shapes

# The Pansu-Wulff shapes

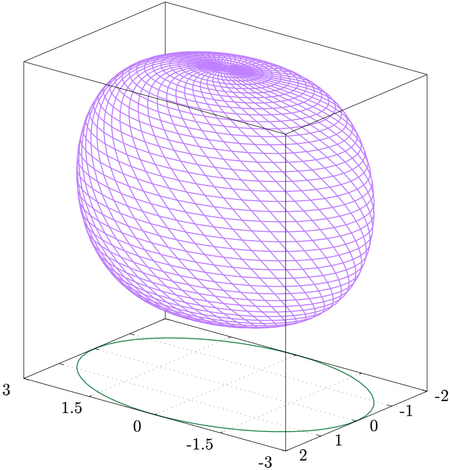



Figure: The Wulff shape associated to the norm $||\cdot||_a = ((x_1/a_1)^2 + (x_2/a_2)^2)^{1/2}$ with $a = (1, 1.5)$. Observe that the projection to the horizontal plane $t = 0$ is an ellipse with semiaxes of lengths 2 and 3.

# The Pansu-Wulff shapes

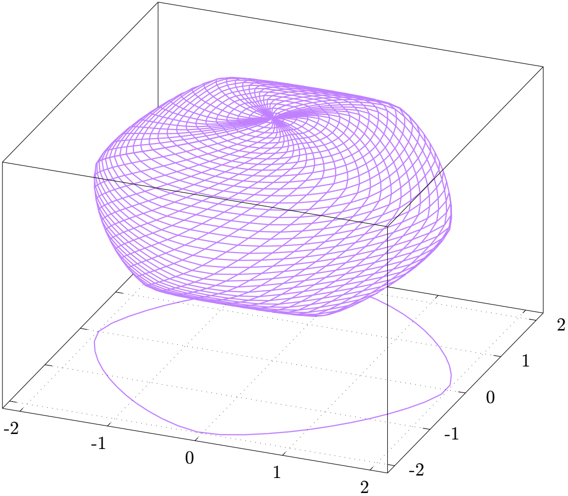



Figure: The Wulff shape $\mathbb{S}_{K_r}$ for the $r$-norm, $r = 1.5$. The horizontal curve is the projection of the equator to the plane $t = 0$. Since the $r$-norm is symmetric, the Wulff shape projects to the set $||\cdot||_r \leqslant 2$ in the $t = 0$ plane.

# The Pansu-Wulff shapes

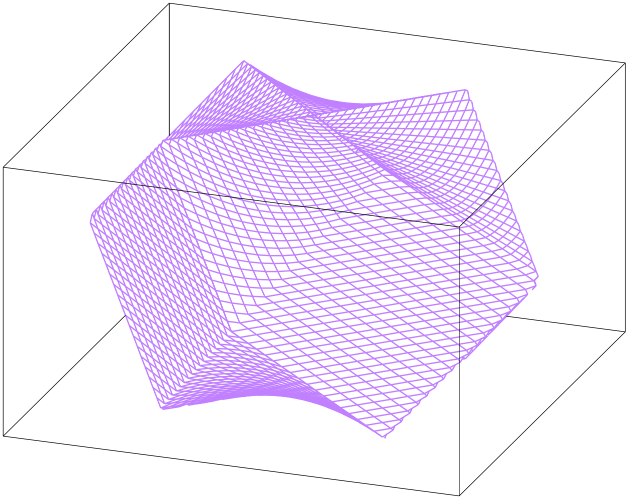


Figure: The ball $\mathbb{B}_1$ obtained as Hausdorff limit of the Wulff shapes $\mathbb{B}_{K_r}$ of the $r$-norm when $r$ converges to 1

# The Pansu-Wulff shapes

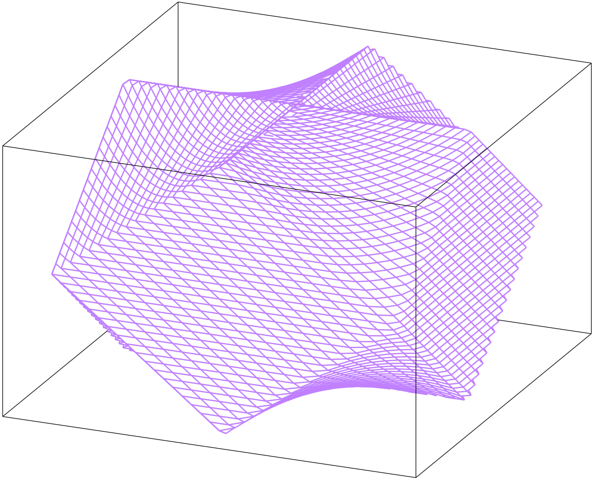


Figure: The ball $\mathbb{B}_\infty$ obtained as Hausdorff limit of the Wulff shapes $\mathbb{B}_{K_r}$ of the $r$-norm when $r$ converges to $\infty$

# The Pansu-Wulff shapes

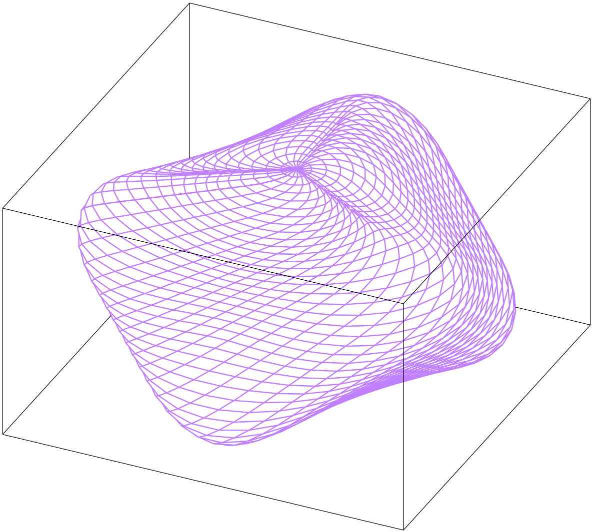


Figure: The Wulff shape $\mathbb{B}_{T,r}$ for the norm $||\cdot||_{T,r}$, with $r = 2$.

# The Pansu-Wulff shapes

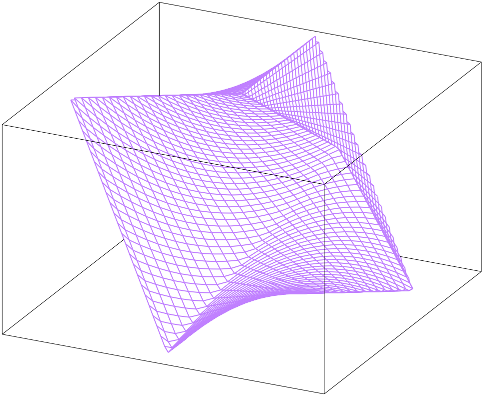


Figure: The ball $\mathbb{B}_T$ obtained as limit of the Wulff shapes $\mathbb{B}_{T,r}$ when $r \to \infty$.

# Pansu-Wulff shapes: minimization properties

(Franceschi et al., arXiv:2007.11384)

Pansu-Wulff boundaries are the only $C^2$ stationary points of area under a volume constraint. Proof after Ritoré-Rosales (2005)

Theorem (Pozuelo-Ritoré, arXiv:2007.04683)

Let $||\cdot||_K$ be the norm associated to an strictly convex body $K \subset \mathbb{R}^2$ with $C^2$ boundary. Let $r > 0$ and $h : rK_0 \to \mathbb{R}$ a $C^0$ function. Consider a subset $E \subset \mathbb{H}^1$ with finite volume and $K$-perimeter such that

$$\text{graph}(h) \subseteq E \subset rK_0 \times \mathbb{R}.$$

Then

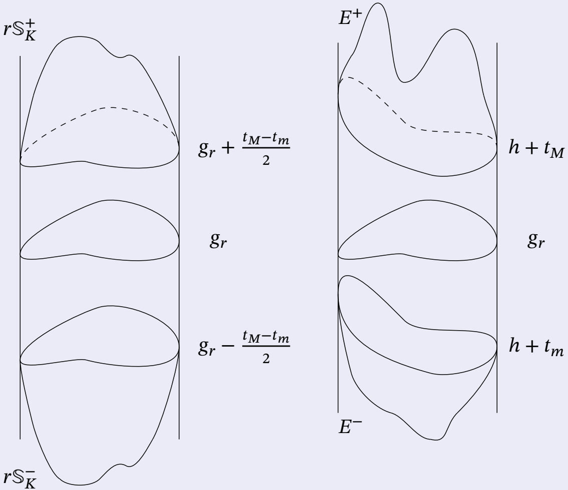
$$P_K(E) \geqslant P_K(\mathbb{B}_E),$$

where $\mathbb{B}_E$ is the Wulff shape with the same volume as $E$.

# Pansu-Wulff shapes: minimization properties

## Proof

We apply a calibration argument



# Regularity properties

## Sets with prescribed mean curvature

Given $f \in C(\Omega)$, $E \subset \Omega$ has prescribed $K$-mean curvature $f$ in $\Omega \Leftrightarrow E$ is a critical point of the functional

$$B \mapsto P_K(B) - \int_{E\cap B} f\, d\mathbb{H}^1, \quad \forall B \subset \Omega \text{ bounded open set} \tag{1}$$

If $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface then (1) $\Leftrightarrow E$ is a critical point of the functional

$$B \mapsto A_K(S \cap B) - \int_{E\cap B} f\, d\mathbb{H}^1, \quad \forall B \subset \Omega \text{ bounded open set}$$

# Regularity of surfaces with prescribed mean curvature

## Regularity of the non-singular set (Giovannardi-Ritoré, 2021)

Let $K$ be a convex body of class $C^2_+$, $\Omega \subset \mathbb{H}^1$ an open set and $E \subset \Omega$ a set of prescribed $K$- mean curvature $f \in C^0(\Omega)$ with Euclidean Lipschitz and $\mathbb{H}$-regular boundary $S$. Then the horizontal curves $\Gamma$ of $S \cap \Omega$ are of class $C^2$.

$\mathbb{H}$-regularity of the boundary means that the surface is locally the level set of a continuous function with non-vanishing continuous horizontal gradient.

## Proof

Quite technical and follows from localizing the first variation along horizontal curves of the surface

# The Bernstein problem

**Bernstein's Theorem (Giovannardi-Ritoré, 2022)**

Let $S \subset \mathbb{H}^1$ be a complete, stable, Euclidean Lipschitz and $\mathbb{H}$-regular surface without singular points. Then $S$ is a vertical plane.

The proof is based on the second variation of the $K$-perimeter

$$\int_S \left( Z(f)^2 + 4\left( Z\left(\frac{\langle N, T\rangle}{|N_h|}\right) - \frac{\langle N, T\rangle^2}{|N_h|^2}\right) f^2\right) \frac{|N_h|}{\kappa(\pi(\nu_h))} dS.$$

We localize the second variation along a horizontal line (where $\kappa(\pi(N_h))$ is constant because $\pi(N_h)$ is constant)

# References

## Recent works


- A.P. Sánchez, Ph.D. Thesis, Tufts U., 2017.
- J. Pozuelo, M. Ritoré, Pansu-Wulff shapes in $\mathbb{H}^1$. Adv. Calc. Var. 16 , no. 1 (2023) 69–98.
- V. Franceschi, R. Monti, A. Righini, and M. Sigalotti, The isoperimetric problem for regular and crystalline norms in $\mathbb{H}^1$. J. Geom. Anal. 33 (2023), no. 1, Paper No. 8, 40 pp.
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- J, Pozuelo, Existence of isoperimetric regions in sub-Finsler nilpotent groups, arXiv:2103.06630.

# References

## Recent works


- G. Giovannardi, M. Ritoré, The Bernstein problem for -Lipschitz surfaces in three-dimensional sub-Finsler Heisenberg groups, Commun. Contemp. Math. (to appear).
- G. Giovannardi, J. Pozuelo, M. Ritoré, Area-minimizing horizontal graphs with low-regularity in the sub-Finsler Heisenberg group , New Trends in Geometric Analysis, RSME Springer Series 10, 2023, pp. 209–226.
- G. Giovannardi, A. Pinamonti, J. Pozuelo, S. Verzellesi, The prescribed mean curvature equation for $t$-graphs in the sub-Finsler Heisenberg group $\mathbb{H}^n$. Adv. Math. 451(2024), Paper No. 109788.

Thanks for your attention