THE ISOPERIMETRIC PROFILE OF COMPACT MANIFOLDS

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In this chapter we introduce the notions of isoperimetric profile and isoperimetric region in an m-dimensional Riemannian manifold M, and describe their basic properties when M is compact. The isoperimetric profile is the function that assigns to any volume the infimum of the perimeter of the sets of this volume, and should be thought of as the best possible isoperimetric inequality in M. Isoperimetric regions are the ones with the smallest possible perimeter for a given volume and are the minimizers of this variational problem.

Right after introducing these concepts, we show existence of isoperimetric sets for any given volume, the continuity and local Hölder continuity of the isoperimetric profile I of M, its positivity and symmetry with respect to $|M|/2$, and we describe the behavior of I for small volumes. Then we focus on regularity properties of the profile by showing a differential inequality satisfied by I^α, for $\alpha \in [1, m/(m-1)]$. This inequality would imply that I^α it is locally the sum of a concave function and a smooth one and hence it enjoys the same regularity properties of concave functions. In particular, it is differentiable once and twice almost everywhere, it has lateral derivatives everywhere and it is an absolutely continuous function. The differential inequality is also very useful for comparison purposes and indeed we use it give a proof, following Bayle, of the classical Levy-Gromov isoperimetric inequality. We also use this inequality to provide a description of the isoperimetric profile of the sphere $\mathbb{S}^m$. Under the hypothesis that the Ricci curvature of the manifold is non-negative we get the concavity of I^α (strict concavity indeed when $1 \leqslant \alpha < m/(m-1)$).

We finish the chapter by looking at the continuity properties of the isoperimetric profile under Lipschitz convergence, as metric spaces, of manifolds.

1. Sets of finite perimeter

Given a Riemannian manifold M and an open subset $\Omega \subset M$, the perimeter of a measurable set $E \subset M$ inside Ω is defined as

$$P(E, \Omega) := \sup \left\{ \int_E \operatorname{div} X \, dM : X \in \mathfrak{X}_0(\Omega), ||X|| \leqslant 1 \right\}. \tag{1}$$

Recall that $\mathfrak{X}_0(\Omega)$ is the space of vector fields with compact support in Ω. When $\Omega = M$, we write $P(E)$ instead of $P(E, M)$.

A measurable set $E \subset M$ has finite perimeter if and only if its characteristic function χ_E is of bounded variation. The reader is referred to Giusti [11] or Maggi [12] for more information on functions of bounded variation.

When E is bounded and has C^1 boundary, the perimeter of E in Ω coincides with the Riemannian measure of $\partial E \cap \Omega$.

A natural topology in the space of measurable sets is given by the L^1-convergence of characteristic functions. We say that a sequence of measurable sets (E_i) converges in $L^1(M)$ to a measurable set E when the characteristic functions χ_{E_i} converge in

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$L^1(M)$ to χ_E. Since $|\chi_{E_i} - \chi_E|$ is the characteristic function of the symmetric difference $E_i \triangle E$, convergence is equivalent to $\lim_{i\to\infty} |E_i \triangle E| = 0$. When the sequence (E_i) L^1-converges to E, we simply write $E = \lim_{i\to\infty} E_i$ or $E_i \to E$.

Two basic properties of sets of finite perimeter are lower semicontinuity and compactness with uniformly bounded perimeter

Proposition 1.1 (Lower semicontinuity of perimeter)**.** *Let $\Omega \subset M$ be an open set in a Riemannian manifold. Let (E_i) a sequence of sets of finite perimeter in Ω converging in $L^1(\Omega)$ to a measurable set E. Then we have*

$$P(E, \Omega) \leqslant \liminf_{i\to\infty} P(E_i, \Omega).$$

Theorem 1.2 (Compactness)**.** *Let $\Omega \subset M$ be a bounded open set with Lipschitz boundary in a Riemannian manifold M. Let (E_i) be a sequence of sets with uniformly bounded perimeters $P(E_i, \Omega)$. Then we can extract a subsequence converging in $L^1(\Omega)$ to a set of finite perimeter $E \subset \Omega$.*

There are other notions of boundary measure in a Riemannian manifold. For smooth submanifolds, one can consider the Riemannian volume element. Another one is the Minkowski content of a set. Given a bounded measurable set E, we consider the tubular neighborhood of E of radius $t > 0$ given by

$$E_t := \{p \in M : d(p, E) \leqslant t\}.$$

Then the Minkowski content of E is defined as

$$M(E) := \liminf_{t\to 0} \frac{|E_t| - |E|}{t}. \tag{2}$$

When E is bounded with C^2 boundary, Steiner's formula implies that the Minkowski content of E equals the Riemannian measure of ∂E. Hence

$$P(E) = M(E)$$

when E is bounded with C^2 boundary. However, it is immediate to check that

$$P(E) \leqslant M(E)$$

for any bounded measurable set E. Indeed, let X be a vector field in M with compact support and $||X|| \leqslant 1$, and let φ_t be its associated flow. Then $\varphi_t(E) \subset E_t$: if $q = \varphi_t(p)$, with $p \in E$, then the curve $s \in [0, t] \to \varphi_s(p)$ connects p and q with length

$$\int_0^t ||X_{\varphi_s(p)}||\, ds \leqslant t.$$

Hence $d(q, E) \leqslant t$. This proves $\varphi_t(E) \subset E_t$. So we have $|\varphi_t(E)| \leqslant |E_t|$ and

$$\int_M \operatorname{div} X\, dM = \frac{d}{dt}\bigg|_{t=0} |\varphi_t(E)| \leqslant \liminf_{t\to 0} \frac{|E_t| - |E|}{t} = M(E).$$

Taking supremum over X, we finally get $P(E) \leqslant M(E)$.

Another notion of boundary measure is the $(m-1)$-dimensional Hausdorff measure, that can be defined in any metric space. We refer the reader to the monographs by Federer [9], Burago, Burago and Ivanov [7] and Evans and Gariepy [8] for definition and properties of the Hausdorff measure.

1.1. **Deformations of sets of finite perimeter.** In isoperimetric problems, it is essential to have ways of modifying the volume of a set while controlling the change of perimeter. We present in this subsection two ways of volume adjustment, one by adding or removing balls and a second one by using the flow associated to a given vector field.

Theorem 1.3 (Volume adjustment using balls)**.** *Let $E \subset M$ be a measurable set in a Riemannian manifold M of finite volume. Given $r > 0$, there exists some $x \in M$ such that*

$$|E \cap B(x,r)| \geqslant \frac{|E|}{2|M|}\, b(r). \tag{3}$$

Proof. Recall that $b(r)$ was defined in (**??**) as $\inf_{y\in M} |B(y,r)|$. The proof follows from an application of Fubini-Tonelli's Theorem to the function $(x,y) \in M \times M \to (\chi_E \chi_{B(y,r)})(x)$. We get

$$\int_E |B(y,r)|\, dM(y) = \int_M |E \cap B(y,r)|\, dM(y),$$

from where we get $|E|\, b(r) \leqslant |M|\, \sup_{y\in M} |E \cap B(y,r)|$ and (3). □

Remark 1.4. Theorem 1.3 implies that, given E, we can find a point $x \in M$ so that the set $E \setminus B(x,r)$ has volume smaller than or equal to $|E|(1 - b(r)/(2|M|)$. By the continuity of the function $r \mapsto |E \setminus B(x,r)|$, removing from E a ball centered at x of suitable radius we can substract from E any volume smaller than $|E|\, b(r)/(2|M|)$.

A similar argument can be applied to the complementary set $E^c = M \setminus E$ to obtain a point $x \in M$ such that $|E^c \cap B(x,r)| \geqslant |E^c| b(r)/2|M|$. In this case, the volume of the set $E \cap B(x,r)$ is larger than or equal to $|E|(1 + b(r)/(2|M|)) + b(r)/2$. So we can add to E any volume between 0 and $|E|b(r)/(2|M|) + b(r)/2$ enlarging the radius of a ball of center x between 0 and r.

Remark 1.5. Given a set $E \subset M$ in a compact connected manifold with $0 < |E| < |M|$, one can obtain a point $x \in M$ so that $|E \cap B(x,r)| > 0$ and $|E^c \cap B(x,r)| > 0$ for any $r > 0$ by the following argument: consider the sets

$$E_0 := \{x \in M : \exists\, r > 0 \text{ with } |E \cap B(x,r)| = 0\},$$

and

$$E_1 := \{x \in M : \exists\, r > 0 \text{ with } |E \cap B(x,r)| = |B(x,r)|\},$$

Then E_0 and E_1 are open sets and $|E \cap E_0| = |E \setminus E_1| = 0$, see Proposition 3.1 in [11]. The set $\tilde{E} := (E \cap E_1) \setminus E_0$ has the same volume as E and $E_1 \subset \operatorname{int}(\tilde{E})$, $E_0 \subset \operatorname{int}(\tilde{E}^c)$. By the connectedness of M and the condition on $|E|$, the boundary of $\tilde{E}$ is not empty. Hence any point $x \in \partial\tilde{E}$ does not belong to $E_0 \cup E_1$.

Theorem 1.6 (Volume adjustment using vector fields)**.** *Let E be a set with finite volume and locally finite perimeter in an open set $\Omega \subset M$, and let $B \subset \Omega$ be an open set such that $P(E,B) > 0$. Then, there exists two constants $C > 0$ and $\overline{m} > 0$ such that, for every $-\overline{m} < m < \overline{m}$, there exists a set $F \subset \Omega$ such that $F = E$ outside B, satisfying*

(1) $|F| = |E| + m$,
(2) $|P(F,\Omega) - P(E,\Omega)| \leqslant C|m|$.

Proof. As $P(E,B) > 0$, there exists a vector field X with compact support in B such that $\int_E \operatorname{div} X\, d\mathcal{H}^n > 0$. The one-parameter group of diffeomorphisms $\{\varphi_t\}_{t\in\mathbb{R}}$

associated to X then satisfies $\frac{d}{dt}\big|_{t=0}|\varphi_t(E)| = \int_E \operatorname{div} X\, d\mathcal{H}^n > 0$. This implies the existence of $\overline{m} > 0$ and an open interval I around 0 such that the function $t \in I \mapsto (|\varphi_t(E)| - |E|) \in (-\overline{m}, \overline{m})$ is a C^1 diffeomorphism. This proves 1.

To prove (ii), we consider the reduced boundary $\partial^* E$ of E. Take $m \in (-\overline{m}, \overline{m})$ and $F = \varphi_{t_m}(E)$ so that $|F| = m$. Then $P(F,\Omega) - P(E,\Omega) = P(F,B) - P(E,B)$. By the area formula

$$|P(F,B) - P(E,B)| = \left| \int_{\partial^* E \cap B} (\operatorname{Jac}(\varphi_{t_m}) - 1)\, d\mathcal{H}^{n-1} \right| \leqslant C' |t_m|\, P(E,B),$$

where C' is a constant depending only on the vector field X. As $|t_m| \leqslant C''|m|$, where $C'' > 0$ is a constant only depending on X and E, we obtain 2. □

2. Regularity of isoperimetric sets

The theory of sets of finite perimeter has been extensively developed in Euclidean spaces and has been recently extended to more general spaces. Some basic results in Euclidean theory can be directly stated and proved in Riemannian manifolds. The techniques in more general metric spaces can be, of course, applied to the metric structure of Riemannian manifolds.

Regularity results for sets minimizing perimeter under a volume constraint were obtained by Morgan, who proves in Corollary 3.8 of [13] the following

Theorem 2.1. *Let E be a measurable set of finite volume minimizing perimeter under a volume constraint in a smooth Riemannian manifold M. Then the boundary of E is the union of a smooth hypersurface S and a singular set S_0 of Hausdorff dimension at most $m-7$.*

When trying to get geometric information on the boundary of a set minimizing perimeter under a volume constraint, the following technical result proved by Sternberg and Zumbrum in Lemma 2.4 of [16] allows to focus just on the regular part S of the boundary.

Lemma 2.2. *Let $E \subset M$ be a minimizer of perimeter under a volume constraint in a smooth Riemannian manifold M. Let S be the regular part of the boundary of E and S_0 its singular part.*

Then for every $\varepsilon > 0$, there exists open sets $U' \subset M$ containing S_0 function and $U \subset M$ contained in an open tubular neighborhood of S_0 of radius ε with $U' \subset\subset M$, and a smooth function $\varphi_\varepsilon : M \to \mathbb{R}$ such that

$$\varphi_\varepsilon(x) = 0 \text{ in } U', \quad \varphi_\varepsilon(x) = 1 \text{ in } M \setminus U,$$

and

$$\int_S |\nabla_S \varphi_\varepsilon|^2 dS \leqslant C\varepsilon,$$

for some constant $C > 0$ depending on E but independent of ε.

3. The isoperimetric profile

Definition 3.1. Let M be a Riemannian manifold. The *isoperimetric profile* of M is the function I_M that assigns, to each $v \in (0, |M|)$, the value

$$I_M(v) := \inf\{P(E) : |E| = v\}. \tag{4}$$

The isoperimetric profile of M will be often denoted simply by I.

Definition 3.2. Let M be a Riemannian manifold. We say that a set $E \subset M$ is *isoperimetric* or that it is an *isoperimetric region* if

$$P(E) = I_M(|E|). \tag{5}$$

If $|E| = v$ we say that E is an isoperimetric region of volume v.

The isoperimetric profile must be understood as an *optimal* isoperimetric inequality on M as, for any subset $F \subset M$ of volume $0 < |F| < |M|$, we have

$$P(F) \geqslant I_M(|F|),$$

with equality precisely for isoperimetric sets. Any function $f : (0, |M|) \to \mathbb{R}^+$ satisfying the inequality $f \leqslant I_M$ provides an isoperimetric inequality on M, namely $P(F) \geqslant f(|F|)$, that it is not optimal in general.

Sometimes, it is convenient to renormalize the isoperimetric profile so that it is defined in a fixed interval (e.g. when we are comparing the profiles of two manifolds or looking at convergence properties of a sequence of isoperimetric profiles).

Definition 3.3. Let M be a compact Riemannian manifold. Its *renormalized isoperimetric profile* $h_M : (0, 1) \to \mathbb{R}^+$ is the function

$$h_M(\lambda) := \frac{I_M(\lambda|M|)}{|M|}. \tag{6}$$

Again we shall often denote h_M simply by h.

If $F \subset M$ then inequality

$$P(F) \geqslant |M|\, h\left(\frac{|F|}{|M|}\right)$$

is satisfied. Equality holds just for isoperimetric regions.

Example 3.4. The classical isoperimetric inequality in Euclidean space states that round balls are the unique isoperimetric sets in $\mathbb{R}^m$. Since the quantity $P(F)/|F|^{(m-1)/m}$ is invariant by Euclidean dilations we have, for any set $F \subset \mathbb{R}^m$ of finite perimeter and volume $|F|$ and any ball $B \subset \mathbb{R}^m$ of volume $|B| = |F|$, the inequality

$$\frac{P(F)}{|F|^{(m-1)/m}} \geqslant \frac{P(B)}{|B|^{(m-1)/m}} = \frac{P(B(0,1))}{|B(0,1)|^{(m-1)/m}} = m\omega_m^{1/m},$$

where $\omega_m = |B(0,1)|$. Hence the isoperimetric profile of the m-dimensional Euclidean space is given by

$$I_{\mathbb{R}^m}(v) = m\omega_m^{1/m} v^{(m-1)/m}. \tag{7}$$

More examples of isoperimetric profiles will be given in the next chapters.

4. Basic properties and continuity

In this section we prove existence of isoperimetric regions in compact manifolds and the continuity of the isoperimetric profile. We prove indeed that the profile is locally Hölder continuous with exponent $(m-1)/m$, where $m = \dim(M)$.

We establish first existence of isoperimetric regions in compact Riemannian manifolds. This follows from compactness and lower semicontinuity of perimeter.

Theorem 4.1. *Let M be a compact Riemannian manifold, and $v \in (0, |M|)$. Then isoperimetric regions of volume v exist on M.*

Proof. We take a minimizing sequence (E_i) of subsets of M with volume v such that $\lim_{i\to\infty} P(E_i) = I_M(v)$. Since $P(E_i)$ is uniformly bounded, we can extract a convergent subsequence to a set E in M of volume v by Theorem 1.2. The lower continuity of the perimeter, Proposition 1.1 implies that

$$I_M(v) \leqslant P(E) \leqslant \liminf_{i\to\infty} P(E_i) = I_M(v).$$

Hence $E \subset M$ is an isoperimetric region of volume v. □

The following result will be of paramount importance

Lemma 4.2 (Relative isoperimetric inequality in balls)**.** *Let M be a Riemannian manifold, $x \in M$ and $r > 0$ such that the exponential map is a diffeomorphism onto $B(x, r)$. Then there exists a constant $K > 0$ such that*

$$P(E, B(x, r)) \geqslant K \min\{|B(x, r) \cap E|, |B(x, r) \setminus E|\}^{(m-1)/m}, \tag{8}$$

for any measurable set $E \subset M$.

Proof. The restriction of the exponential map to the ball in T_xM centered at the origin of radius $r > 0$ is a bi-Lipschitz map to $B(x, r)$. Then we transfer the relative isoperimetric inequality in the Euclidean ball [6] to $B(x, r)$. □

Remark 4.3. The constant K in (8) depends on x and r. Under appropriate conditions on the manifold, this constant can be taken uniform for $x \in M$ and r smaller than some fixed $r_0 > 0$.

Let us show now that the isoperimetric profile is a positive symmetric function that extends continuously to $v = 0$ and $v = |M|$.

Lemma 4.4. *The isoperimetric profile of a compact Riemannian manifold is a positive function in the interval $(0, |M|)$, symmetric with respect to $|M|/2$ and extends continuously to 0 at the endpoints of the interval.*

Proof. A standard argument in measure theory (see Proposition 3.1 in Giusti's monograph [11]) implies the existence of $x \in M$ and $r > 0$ smaller than the injectivity radius of M so that $|E \cap B(x, r)|$, $|B(x, r) \setminus E| > 0$. We can then apply the relative isoperimetric inequality (8) in $B(x, r)$ to conclude that $P(E) \geqslant P(E, B(x, r)) > 0$.

The symmetry of the profile follows because if $E \subset M$ is an isoperimetric region of volume v, then $M \setminus E$ is an isoperimetric region of volume $|M| - v$ with the same perimeter as E.

To prove that I extends continuosly as 0 at the endpoints of the interval, it is enough to do it at $v = 0$ by the symmetry property. We simply fix a point $x \in M$ and consider the function $r \mapsto |B(x, r)|$, that is continuous, increasing, and approaches 0 when $r \to 0^+$. The perimeters $P(B(x, r))$ also converge to 0 when $r \to 0^+$, so that we have $P(B(x, r)) \geqslant I(|B(x, r)|)$. Letting $r \to 0^+$ we get the result. □

Let us now prove the continuity of the isoperimetric profile of M.

Theorem 4.5. *The isoperimetric profile of a compact Riemannian manifold is a continuous function. Moreover, it is locally Hölder continuous with exponent $(m - 1)/m$ in the interval $(0, |M|)$.*

Proof. We give two proofs of this result. For the first one, that has the advantage of proving the local Hölder continuity of I, we follow the arguments by Gallot, see Lemme 6.2 in [10].

First proof. Fix some $v_0 > 0$ small and take $0 < v < v_0$ close enough to v_0, so that we can choose $0 < r < r_0$ satisfying

$$\frac{b(r)}{2|M|}\, v_0 = v_0 - v,$$

where $b(r) = \inf_{x\in M} |B(x,r)|$, already defined, is continuous, and $r_0 > 0$ is a uniform radius in M so that we have

$$P(B(x,s)) \leqslant C_n s^{n-1}, \quad b(s) \geqslant C'_n s^n, \quad 0 < s < r_0,$$

by Rauch comparison theorems. Here C_n, $C'_n > 0$ are n-dimensional constants.

Fix $\varepsilon > 0$ and take a finite perimeter set $E \subset M$ of volume v_0 such that $P(E) \leqslant I(v_0) + \varepsilon$. Then we have

$$\int_M |B(x,r) \cap E|\, dM(x) = \int_E |B(x,r)|\, dM(x). \tag{9}$$

This formula is obtained by applying Fubini-Tonelli's Theorem to the function $(x,y) \in M \times M \mapsto \chi_{B(x,r)}\chi_E(y)$. Inequality (9) implies the existence of $x \in M$ such that

$$|B(x,r) \cap E| \geqslant \frac{b(r)}{2|M|}\, |E|.$$

and so $v_0 - v \leqslant |B(x,r) \cap E|$ by the choice of r. From the continuity of the non-decreasing function $s \mapsto |B(x,r) \cap E|$, we can find $s \in (0,r]$ such that $|B(x,s) \cap E| = v_0 - v$. Hence $|E \setminus B(x,s)| = v$ and we get

$$I(v) \leqslant P(E \setminus B(x,s)) \leqslant P(E) + P(B(x,s)) \leqslant I(v_0) + \varepsilon + C\left(\frac{v_0 - v}{v_0}\right)^{(m-1)/m},$$

for some constant $C > 0$ depending on n and $|M|$. As $\varepsilon > 0$ is arbitrary we get

$$I(v) \leqslant I(v_0) + C\left(\frac{v_0 - v}{v_0}\right)^{(m-1)/m}.$$

Considering the volumes $|M| - v_0$, $|M| - v$, the symmetry of the isoperimetric profile implies a similar inequality for $v > v_0$. Hence we get

$$|I(v) - I(v_0)| \leqslant C\left|\frac{v_0 - v}{v_0}\right|^{(m-1)/m},$$

for any pair of volumes v, v_0 close enough so that $b(r)/(2|M|)v_0 = |v_0 - v|$ is satisfied for some $0 < r < r_0$.

Second proof. Take a sequence $(v_i)_{i\in\mathbb{N}}$ of volumes satisfying $0 < v_i < |M|$ converging to $0 < v < |M|$. For each i, take an isoperimetric set E_i of volume v_i. As the perimeters $P(E_i)$ are uniformly bounded, we can take a convergent subsequence to some set $E \subset M$ of finite perimeter and volume v. By the lower semicontinuity of perimeter we get

$$I(v) \leqslant P(E) \leqslant \liminf_{i\to\infty} P(E_i) = \liminf_{i\to\infty} I(v_i).$$

This shows that I is lower semicontinuous.

Let us prove now the upper semicontinuity of I. Take an isoperimetric set $E \subset M$ of volume. By the Deformation Theorem 1.6, we can find, for large i, sets E_i of volume v_i and finite perimeter so that

$$I(v_i) \leqslant P(E_i) \leqslant P(E) + C|v - v_i| \leqslant I(v) + C|v - v_i|.$$

Taking $\limsup$ we obtain the upper semicontinuity of perimeter. ∎

5. Asymptotic expansion for small volumes

The following result ensures that the isoperimetric profile of a compact manifold for small volumes is asymptotically the one of the Euclidean space of the same dimension. We closely follow the direct proof by Berard and Meyer, see Theorem in Appendix C of [5].

Theorem 5.1. *Let M be a compact Riemannian manifold. For every $\varepsilon > 0$, there exists a positive constant $v_0 = v_0(M, g, \varepsilon)$ such that any set $E \subset M$ of volume $0 < v \leqslant v_0$ satisfies*

$$P(E) \geqslant (1-\varepsilon)c(m)|E|^{(m-1)/m}, \tag{10}$$

where $c(m) = m\omega_m^{1/m}$ is the isoperimetric constant in $\mathbb{R}^m$.

Proof. Let $\rho > 0$ so that, for any set F contained in a geodesic ball in M of radius 2ρ, we have

$$P(F) \geqslant (1-\varepsilon/2)c(m)|F|^{(m-1)/m}.$$

The existence of ρ follows because the Riemannian metric on M is uniformly and asymptotically Euclidean.

Let $x_1, \ldots, x_\ell$ be a maximal family of points in M so that the balls $B(x_i, \rho/2)$ are disjoint. By the coarea formula, for each i we have

$$|E \cap B(x_i, 2\rho)| \geqslant \int_\rho^{2\rho} A(\partial B(x_i, r) \cap E)\, dr,$$

and so there exists $\rho(i) \in [\rho, 2\rho]$ such that

$$A(\partial B(x_i, \rho(i)) \cap E) \leqslant \frac{|E|}{\rho}.$$

Let $\mathcal{B}$ be the set of connected components of $M \setminus \bigcup_{i=1}^{\ell} B(x_i, \rho(i))$, and E' the disjoint union of the sets $E \cap F$, where $F \in \mathcal{B}$. Then we have

$$P(E') \leqslant P(E) + 2\ell \frac{|E|}{\rho}$$

and, since each component of E' is contained in a ball of radius 2ρ we obtain, using the concavity of the function $s \mapsto s^{(m-1)/m}$

$$\begin{aligned}(1-\varepsilon/2)c(m)|E|^{(m-1)/m} &= (1-\varepsilon/2)c(m)\bigg(\sum_{F\in\mathcal{B}} |E \cap F|\bigg)^{(m-1)/m} \\ &\leqslant (1-\varepsilon/2)c(m) \sum_{F\in\mathcal{B}} |E \cap F|^{(m-1)/m} \\ &\leqslant P(E') \leqslant P(E) + 2\ell\frac{|E|}{\rho}.\end{aligned}$$

Hence

$$\frac{P(E)}{|E|^{(m-1)/m}} \geqslant (1-\varepsilon/2)c(m) - 2\ell\frac{|E|^{1/m}}{\rho}.$$

So it is enough to take

$$|E| \leqslant \bigg(\frac{\varepsilon c(m)\rho}{4\ell}\bigg)^{1/m}.$$

to obtain (10). □

Remark 5.2. Using small balls centered at some fixed point of M to get an upper bound for the isoperimetric profile, we get from Theorem 5.1

$$\lim_{v\to 0} \frac{I(v)}{v^{m/(m-1)}} = m\omega_m. \tag{11}$$

From this inequality we obtain

$$\lim_{\lambda\to 0} \frac{h(\lambda)}{\lambda^{m/(m-1)}} = m\omega_m |M|^{1/(m-1)}. \tag{12}$$

6. Differential properties

In this section we prove concavity properties of some powers of the isoperimetric profile of a compact manifold. In particular, we show that the power $m/(m-1)$ of the isoperimetric profile I of an m-dimensional compact Riemannian manifold M is locally the sum of a concave function and a smooth function. Hence the isoperimetric profile satisfies all regularity properties of concave functions such as existence of side derivatives everywhere and existence of second derivative almost everywhere. We prove this result using deformations of the regular part of an isoperimetric boundary, so that it is necessary to use the full regularity theory for isoperimetric minimizers, to derive a differential inequality satisfied in a weak sense by the isoperimetric profile. We follow the proof of Proposition 3.3 by Morgan and Johnson [14], based on an argument by Bavard and Pansu [1]. See also Bayle [2] and Bayle and Rosales [4]. The continuity of the isoperimetric profile proved in the previous section is an essential ingredient in the proof.

Once we have obtained the differential inequality for the profile, we use it to give another proof of Levy-Gromov isoperimetric inequality for compact manifolds with a positive lower bound on its Ricci curvature, following the proof given by Bayle in his Ph.D. Thesis.

First we prove the following elementary result for concave functions.

Lemma 6.1. *Let $f : I \to \mathbb{R}$ be a continuous function defined on an open interval $I \subset \mathbb{R}$. Assume that for all $x \in I$ there is a family of smooth functions $(f_{x,\varepsilon})_{\varepsilon>0}$, each one defined in a neighborhood of x, such that $f \leqslant f_{x,\varepsilon}$, $f(x) = f_{x,\varepsilon}(x)$, and $\limsup_{\varepsilon\to 0} f''_{x,\varepsilon}(x) \leqslant 0$. Then f is a concave function.*

Proof. If f is not concave, then there exists $\delta > 0$ such that the function $f_\delta(x) := f(x) - \delta x^2$ is not concave. To see this, represent the subgraph of f as the closure of the union of the increasing family (when $\delta \to 0$) of the subgraphs of f_δ. If the subgraph of f is not a convex set, then some of the subgraphs of f_δ are not convex sets.

As f_δ is not concave, there exist two points $x_1 < x_2$ on I such that the function $L(x) - f_\delta(x)$ has a positive maximum $x_0 \in (x_1, x_2)$. Here $L(x)$ is the linear function passing through $(x_1, f_\delta(x_1))$ and $(x_2, f_\delta(x_2))$. Then each one of the smooth functions $L(x) - f_{x_0,\varepsilon}(x) + \delta x^2$ has a maximum at x_0. Hence $f''_{x_0,\varepsilon}(x_0) \geqslant 2\delta > 0$ for all $\varepsilon > 0$, contradicting the hypothesis $\limsup_{\varepsilon\to 0} f''_{x_0,\varepsilon}(x) \leqslant 0$. □

Let us now give sense to the inequality $f'' \leqslant C$ for non-smooth functions.

Definition 6.2. Let $f : (a,b) \to \mathbb{R}$ be a continuous function and $x_0 \in (a,b)$. Let $C \in \mathbb{R}$. We say that $f''(x_0) \leqslant C$ *in weak sense* if there exists a sequence of smooth function (f_i), each one defined on an interval containing x_0, so that

(1) $f \leqslant f_i$ in the common domain of definition,
(2) $f(x_0) = f_i(x_0)$,

(3) $\limsup_{i\to\infty} f_i''(x_0) \leqslant C$.

We prove now the main result in this section

Theorem 6.3. *Let M be an m-dimensional compact Riemannian manifold, and I its isoperimetric profile. Let $1 \leqslant \alpha \leqslant m/(m-1)$. Then I^α satisfied the differential inequality*

$$(I^\alpha)'' \leqslant K_0 (I^\alpha)^{(\alpha-2)/\alpha}, \tag{13}$$

in weak sense in $(0,|M|)$, where $K_0 = -\alpha \inf_{|v|=1} \operatorname{Ric}(v,v)$. Equality holds in (13) for some $v_0 \in (0,|M|)$ if $\alpha = m/(m-1)$, the Ricci curvature of M is equal to $-K_0/\alpha$ and there exists an isoperimetric region in M with totally umbilical boundary.

Moreover, I^α is locally the sum of a concave function and a smooth function. In particular, I^α has side derivatives everywhere, it is differentiable and has second derivatives almost everywhere, and it is absolutely continuous.

Proof. To prove the concavity property of I^α we use Lemma 6.1. We fix some $0 < v_0 < |M|$ and take an isoperimetric region $E \subset M$. We let S be the regular part of its boundary and S_0 the singular set. By Lemma **??**, there exists a sequence of smooth functions (f_i) with compact support in S and satisfying

(1) $0 \leqslant f_i \leqslant 1$ for all i,
(2) the sequence (f_i) is non-decreasing and pointwise converges to the constant function 1 on S.
(3) $\lim_{i\to\infty} \int_S |\nabla f_i|^2 dS = 0$.

For any i, we take a vector field X_i with compact support on M so that $X_i = f_i N$ on S, where N is the outer unit normal to E on S. Let $\{\varphi_t^i\}_{t\in\mathbb{R}}$ the associated flow. We have

$$\frac{d}{dt}\bigg|_{t=0} |\varphi_t^i(E)| = \int_S f_i \, dS > 0,$$

and so we can take the volume as a parameter of the deformation for v close to v_0 and write $A_i(v) = P(\varphi_{t(v)}^i(E))$. We trivially have $A_i^\alpha(v) \geqslant I^\alpha(v)$, with equality at v_0. Let us compute the second derivative of A_i^α with respect to v at $v = v_0$. First we have, for $A = A_i$,

$$\frac{d}{dv} A^\alpha = \alpha A^{\alpha-1} \frac{dA/dt}{dv/dt}.$$

We take a second derivative to obtain

$$\frac{d^2}{dv^2} A^\alpha = \alpha A^\alpha \bigg((\alpha-1) \frac{1}{A} \bigg(\frac{dA/dt}{dv/dt} \bigg)^2 + \frac{1}{(dv/dt)^2} \bigg(\frac{d^2 A}{dt^2} - \frac{dA/dt}{dv/dt} \frac{d^2 v}{dt^2} \bigg) \bigg)$$

We evaluate this derivative when $A = A_i$ at $v = v_0$ to get

$$\frac{d^2}{dv^2} A_i^\alpha(v_0) = \alpha A(S)^\alpha \bigg((\alpha-1) \frac{H^2}{A(S)} + \frac{1}{\big(\int_S f_i dS\big)^2} \bigg(\int_S \big(|\nabla f_i|^2 - (\operatorname{Ric}(N,N) + |\sigma|^2) f_i^2 \, dS \bigg) \bigg),$$

since $(dA/dt)/(dv/dt)$ at $t = 0$ is equal to the constant mean curvature H of S and $A'' - HV''$, where $'$ denotes the derivative with respect to t, equals the second variation operator (??). Taking lim sup when $i \to \infty$ we get

$$\begin{aligned}\limsup_{i\to\infty} \frac{d^2}{dv^2} A_i^\alpha(v_0) &= -\alpha \frac{A(S)^\alpha}{A(S)^2} \times \\ &\qquad \times \int_S \left(\operatorname{Ric}(N,N) + \left(|\sigma|^2 - (\alpha-1)H^2 \right) \right) dS \\ &\leqslant -\alpha A(S)^{\alpha-2}, \inf_S \operatorname{Ric}(N,N) \\ &\leqslant -\alpha (I(v_0)^\alpha)^{(\alpha-2)/\alpha} K_0\end{aligned}$$

since

$$|\sigma|^2 - (\alpha-1)H^2 \geqslant |\sigma|^2 - \frac{H^2}{m-1} \geqslant 0$$

by our hypothesis on α and inequality $k\sum_{i=1}^k a_i^2 \geqslant \left(\sum_{i=1}^k a_i\right)^2$.

To prove the concavity property of I^α we use Lemma 6.1. For $0 < v_1 < v_2 < |M|$, the continuity of the isoperimetric profile implies the existence of a constant C such that

$$-\alpha I(v_0)^{\alpha-2} \left\{ \inf_{S,|e|=1} \operatorname{Ric}(e,e) \right\} \leqslant C, \tag{14}$$

for all $0 < v_1 < v < v_2$. This implies, by Lemma 6.1, that $I^\alpha - Cv^2$ is a concave function.

The differentiability properties, as well as the absolute continuity, follow from the corresponding ones for concave functions, see [15, § 24]. □

Remark 6.4. It is immediate to check that the $m/(m-1)$ power of the renormalized isoperimetric profile h satisfies the same differential inequality as $I^{m/(m-1)}$

$$(h^{m/(m-1)})'' \leqslant K_0 (h^{m/(m-1)})^{-(m-2)/m}, \tag{15}$$

where $K_0 = -(m/(m-1)) \inf_{|v|=1} \operatorname{Ric}(v,v)$. Equality holds in (15) for some $\lambda_0 \in (0,1)$ if the Ricci curvature of M is equal to $-(m-1)K_0/m$ and there exists an isoperimetric region in M of volume $\lambda_0|M|$ with totally umbilical boundary.

Corollary 6.5. *Let M be a compact Riemannian manifold with non-negative Ricci curvature. Then, for all $1 \leqslant \alpha \leqslant m/(m-1)$, the function I^α is concave. In particular, the isoperimetric profile is a strictly concave function.*

Proof. It follows from the proof of Theorem 6.3 simply by taking the constant C in (14) equal to 0. □

The strict concavity of the isoperimetric profile is an important property related to the connectedness of isoperimetric regions in M.

Theorem 6.6. *Let M be a compact Riemannian manifold and assume that I is strictly concave. Then isoperimetric regions in M are connected. In particular, the hypothesis is satisfied when M has non-negative Ricci curvature.*

Proof. Assume that an isoperimetric region $E \subset M$ of volume v has two connected components E_1, E_2 of volumes $v_1 > 0$, $v_2 > 0$, respectively. Then we have

$$I(v_1) + I(v_2) \leqslant P(E_1) + P(E_2) = P(E) = I(v).$$

But this is a contradiction since for the concave function I satisfying $I(0) = 0$, we should have $I(v) < I(v_1) + I(v_2)$ when $v = v_1 + v_2$ and $v_1, v_2 > 0$. □

Theorem 6.7. *Let M be a compact Riemannian manifold and let I be its isoperimetric profile.*

(1) *If I is regular at v_0 then all isoperimetric regions have boundary mean curvature equal to $I'(v_0)$.*
(2) *If I is not regular at v_0, there exists two isoperimetric regions in M of volume v_0 with boundary mean curvatures $I'_+(v_0)$ and $I'_-(v_0)$.*

Proof. For the first assertion, we take an isoperimetric region $E \subset M$ of volume v_0. We take the deformation $\{E_t\}_{t\in\mathbb{R}}$ associated to any vector field X with compact support in M such that $\operatorname{supp}(X) \cap S_0 = \emptyset$ and $\int_E \operatorname{div} X\, dM \neq 0$. Then we can express this deformation taking the volume as a parameter to get a function $A(v)$ whose derivative at $v = v_0$ is the mean curvature H of the boundary of E. Since $I \leqslant A$ and $I(v_0) = A(v_0)$, and I is regular at v_0, we get $I'(v_0) = A'(v_0) = H$.

To prove the second assertion, we take a sequence (v_i) of regular values of I so that v_i decreases to v_0. We take a sequence of isoperimetric regions (E_i) of volumes v_i and boundary mean curvature H_i. Then E_i converges in the L^1 topology to a set $E \subset M$ of volume v that is isoperimetric. We have

$$H = \lim_{i\to\infty} H_i = \lim_{i\to\infty} I'(v_i) = I'_+(v_0).$$

Analogously, we can construct an isoperimetric region with boundary mean curvature $I'_-(v_0)$. □

We end this section with a proof of the Levy-Gromov inequality using the differential inequality (15). For the most part of the proof we follow Bayle [3].

Theorem 6.8 (Levy-Gromov inequality)**.** *Let M be a compact Riemannian manifold satisfying*

$$\operatorname{Ric} \geqslant (m-1)\,\kappa_0,$$

for some $\kappa_0 > 0$. Then we have

$$h(\lambda) \geqslant h_0(\lambda),$$

for any $\lambda \in (0,1)$, where h_0 is the isoperimetric profile of the sphere $\mathbb{S}_{\kappa_0}$ of constant sectional curvature κ_0. If equality holds for some $\lambda_0 \in (0,1)$, then M is isometric to $\mathbb{S}_{\kappa_0}$.

Proof. Let $f = h^{m/(m-1)}$ and $f_0 = h_0^{m/(m-1)}$. Then $f'' \leqslant g(f)$ in weak sense, where $g(x) = -(m\kappa_0/(m-1))x^{-(m-2)/m}$ by Remark 6.4. We also have that $f_0'' = g(f_0)$, where $f_0 = h_0^{m/(m-1)}$.

Assume by contradiction that there exists $\lambda_0 \in (0,1)$ so that $f(\lambda_0) < f_0(\lambda_0)$. Take a maximal interval $J \subset [0,1]$ containing λ_0 so that $f < f_0$ in the interior of J. Then we have $f_0 - f > 0$ in the interior of J and

$$(f_0 - f)'' \geqslant g(f_0) - g(f) > 0$$

in the interior of J by the strict monotonicity of g. But this not possible since $f - f_0$ would be a strictly positive strictly convex function in the interior of J vanishing at the end points of the interval. This shows that $f \geqslant f_0$ and so $h \geqslant h_0$.

Let us now discuss the equality case. Assume the existence of $\lambda_0 \in (0,1)$ such that $f(\lambda_0) = f_0(\lambda_0)$. By the symmetry of the renormalized isoperimetric profiles with respect to $\lambda = \frac{1}{2}$, we may assume that $\lambda_0 \in (0, \frac{1}{2}]$. Let us distinguish two cases:

Case 1. If $\lambda_0 < \frac{1}{2}$, we take any $\varepsilon > 0$ so that $\lambda_0 + \varepsilon < \frac{1}{2}$ and define

$$f_0^\varepsilon(\lambda) := f_0(\lambda + \varepsilon) - f_0(\lambda_0 + \varepsilon) + f_0(\lambda_0).$$

The function f_0^ε is obtained translating the graph of f_0 to the left a distance ε and then translating it down so that $f_0^\varepsilon(\lambda_0) = f_0(\lambda_0) = f(\lambda_0)$. We have the following properties

- $f_0^\varepsilon(\lambda_0) = f(\lambda_0)$,
- $f_0^\varepsilon > f$ in $(\lambda_0 - \delta, \lambda_0)$ for some small $\delta > 0$,
- $f_0^\varepsilon(0) > 0 = f(0)$,
- $(f_0^\varepsilon)'' \geqslant g(f_0^\varepsilon)$,

The first property follows from the definition of f_0^ε. For the second one we observe that $f'_-(\lambda_0) = f'_0(\lambda_0) > f'_0(\lambda_0+\varepsilon) = (f_0^\varepsilon)'(\lambda_0)$. For the third one, the strict concavity of f_0 implies $f_0(\lambda_0+\varepsilon) - f_0(\lambda_0) < f_0(\varepsilon)$ since $f_0(0) = 0$. For the last property, we compute

$$(f_0^\varepsilon)''(\lambda) = f_0''(\lambda+\varepsilon) = g(f_0(\lambda+\varepsilon)) \geqslant g(f_0^\varepsilon(\lambda)),$$

the last inequality since $f_0(\lambda+\varepsilon) > f_0^\varepsilon(\lambda)$, an inequality equivalent to $f_0(\lambda_0+\varepsilon) > f_0(\lambda_0)$.

We claim that $f_0^\varepsilon \geqslant f$ in the interval $(0, \lambda_0]$. Otherwise there exists some $\mu_0 \in (0, \lambda_0)$ such that $f_0^\varepsilon - f > 0$ in (μ_0, λ_0) and $(f_0^\varepsilon - f)$ vanishes at the end points of the interval. But $(f_0^\varepsilon - f)'' \leqslant g(f_0^\varepsilon) - g(f) \geqslant 0$, a contradiction.

So we have $f_0^\varepsilon \geqslant f$ in the interval $(0, \lambda_0]$ for all $\varepsilon > 0$ such that $\lambda_0 + \varepsilon < \frac{1}{2}$. Letting $\varepsilon \to 0$, we get $f_0 \geqslant f$ and, since $f \geqslant f_0$ we obtain $f = f_0$ in the interval $[0, \lambda_0]$. By the asymptotic expansion for h, h_0 at $\lambda = 0$, we get $|M| = |\mathbb{S}_{\kappa_0}|$. We conclude that M is isometric to $\mathbb{S}_{\kappa_0}$ by the rigidity part of Bishop's volume comparison theorem, see Theorem xx in [].

Case 2. Assume that $\lambda_0 = \frac{1}{2}$. Consider, for $\varepsilon > 0$ small enough, the function

$$f_0^\varepsilon(\lambda) := f_0(\lambda+\varepsilon) + f(\tfrac{1}{2} - \varepsilon) - f_0(\tfrac{1}{2}).$$

The following properties can be obtained using arguments similar to the ones in Case 1.

- $f_0^\varepsilon(\frac{1}{2} - \varepsilon) = f(\frac{1}{2} - \varepsilon)$.
- $f_0^\varepsilon > f$ in $(\frac{1}{2} - \varepsilon - \delta, \frac{1}{2} - \varepsilon)$ for some $\delta > 0$,
- $f_0^\varepsilon(0) > 0 = f(0)$,
- $(f_0^\varepsilon)'' \geqslant g(f_0^\varepsilon)$

Now we reason as the first case to conclude that $f_0^\varepsilon \geqslant f$ in the interval $(0, \frac{1}{2} - \varepsilon]$. Letting $\varepsilon \to 0$ we get $f_0 \geqslant f$ in $(0, \frac{1}{2}]$ and so $f_0 = f$ in $[0, \frac{1}{2}]$. We conclude the proof as in the previous case. □

Remark 6.9. The techniques used allow to prove that the isoperimetric regions in the sphere $\mathbb{S}^m_\kappa$ are the geodesic balls, bounded by totally umbilical spheres with constant mean curvature. This follows easily by taking the renormalized isoperimetric profile $f = h^{m/(m-1)}$ of $\mathbb{S}^m_\kappa$, that satisfies the differential inequality $f'' \leqslant g(f)$, where

$$g(x) = -\frac{m}{m-1}\kappa\, x^{-(m-2)/m},$$

We also take the relative isoperimetric profile of geodesics spheres $f_0 = h_0^{m/(m-1)}$, that satisfies the differential equation $f_0'' = g(f_0)$. We obviously have

$$f_0 - f \geqslant 0,$$

and also

$$(f_0 - f)'' \geqslant g(f_0) - g(f) \geqslant 0,$$

what implies $f = f_0$.

References

[1] C. Bavard and P. Pansu. Sur le volume minimal de $\mathbf{R}^2$. *Ann. Sci. École Norm. Sup. (4)*, 19(4):479–490, 1986.

[2] V. Bayle. A differential inequality for the isoperimetric profile. *Int. Math. Res. Not.*, (7):311–342, 2004.

[3] V. Bayle. *Propriétés de Concavité du Profil Isopérimétrique et Applications*. PhD thesis, Institut Fourier, 2004.

[4] V. Bayle and C. Rosales. Some isoperimetric comparison theorems for convex bodies in Riemannian manifolds. *Indiana Univ. Math. J.*, 54(5):1371–1394, 2005.

[5] P. Bérard and D. Meyer. Inégalités isopérimétriques et applications. *Ann. Sci. École Norm. Sup. (4)*, 15(3):513–541, 1982.

[6] J. Bokowski and J. E. Sperner. Zerlegung konvexer Körper durch minimale Trennflächen. *J. Reine Angew. Math.*, 311/312:80–100, 1979.

[7] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[8] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[9] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[10] S. Gallot. Inégalités isopérimétriques et analytiques sur les variétés riemanniennes. *Astérisque*, (163-164):5–6, 31–91, 281 (1989), 1988. On the geometry of differentiable manifolds (Rome, 1986).

[11] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[12] F. Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[13] F. Morgan. Regularity of isoperimetric hypersurfaces in Riemannian manifolds. *Trans. Amer. Math. Soc.*, 355(12):5041–5052, 2003.

[14] F. Morgan and D. L. Johnson. Some sharp isoperimetric theorems for Riemannian manifolds. *Indiana Univ. Math. J.*, 49(3):1017–1041, 2000.

[15] R. T. Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

[16] P. Sternberg and K. Zumbrun. On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint. *Comm. Anal. Geom.*, 7(1):199–220, 1999.