Isoperimetric inequalities in unbounded convex bodies

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Problem: Minimize the relative perimeter under a volume constraint inside a convex body

$C \subset \mathbb{R}^n$ *convex body* (closed convex set with non-empty interior; no regularity assumption on the boundary; either bounded or unbounded)

The *relative perimeter* of $E \subset C$ is defined by

$$P_C(E) := \sup \left\{ \int_E \operatorname{div} X : X \in \mathfrak{X}_0^1(\operatorname{int}(C)), |X| \leq 1 \right\}.$$

$P_C(E) = P(E, \operatorname{int}(C))$

The *volume* of E is denoted by $|E|$

Problem: Given $0 < v < |C|$, compute $\inf\{P_C(E) : |E| = v\}$ and study the minimizers

Isoperimetric profile and isoperimetric regions

The *isoperimetric profile* of C is the function $I_C : (0, |C|) \to \mathbb{R}^+$

$$I_C(v) := \inf\{P_C(E) : |E| = v\}$$

A set $E \subset C$ is *isoperimetric* if $I_C(|E|) = P_C(E)$.

I_C determines an isoperimetric inequality in C: for any $F \subset C$ we have

$$P_C(F) \geq I_C(|F|)$$

with equality iff F is isoperimetric. I_C can be thought of as the *optimal isoperimetric inequality* on C

Problems

1. Existence and regularity of isoperimetric sets
2. Geometric and topological properties of isoperimetric regions
3. Determination and properties of the isoperimetric profile. Is it positive, regular, absolutely continuous?

Comments

- Existence is not guaranteed in unbounded sets
- Although interior regularity is established, boundary regularity results have not been obtained for non-smooth ∂C (e.g. polyhedra)
- Symmetrization does not work even in highly symmetric cases
- Only the case of the ball is explicitly solved

Problem related to:

1. Van der Waals-Cahn-Hilliard theory of phase transitions (Modica, Sternberg,Pacard-Ritoré,...)
2. Capillarity problems (Finn)
3. Shape of A/B block copolymers separated with distinct phases (Thomas et al., Ohta-Kawasaki)
4. Sobolex-Poincaré, Faber-Krahn and Cheeger inequalities

Results known in the bounded case

1. Isoperimetric profile of the ball (Bokowski-Sperner, Burago-Zalgaller)
2. Isoperimetric regions exist and the profile is positive and symmetric. It is asymptotically the one of the half-plane when the boundary is smooth
3. $\left(\frac{n}{n-1}\right)$-concavity of the profile (Kuwert in the smooth case, E. Milman in the general one; also proved by approximation in Hausdorff distance by Ritoré-Vernadakis)
4. Connectedness of isoperimetric boundaries and regions for $C^{2,\alpha}$ boundary, (Stredulinsky-Ziemer, Ritoré-Vernadakis)

Plan of the talk

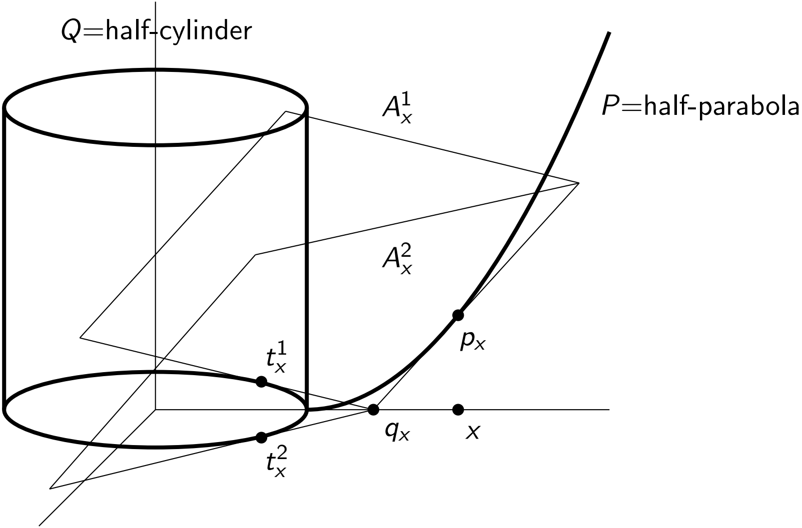
Goal

- Extend, if possible, known results for the bounded case to the unbounded one

Contents

1. Positivity of the profile (convex bodies of uniform geometry and asymptotic cylinders)
2. Generalized existence
3. $\left(\frac{n}{n-1}\right)$-concavity of the profile
4. Small volumes

Positivity of I_C. Is $I_C > 0$?



Take $C = \mathrm{conv}(Q \cup P)$

First observe that $I_C \leq I_{C_p}$ $\forall p \in \partial C$ (C_p is the tangent cone to C at p), and

$$I_{C_p}(v) = n\alpha_p v^{(n-1)/n},$$

where $\alpha_p = |B_{C_p}(p, 1)|$ measures the aperture of the cone ($B_C(p, r) := C \cap B(p, r)$ is the relative ball). In the previous example

$$\lim_{x\to\infty} \alpha_{p_x} = \lim_{x\to\infty} |B_{C_{p_x}}(p_x, 1)| = 0.$$

Hence $I_C \equiv 0$.

Observe that in this example

$$\inf_{p\in C} |B_C(p, r)| \leq \inf_x |B_C(p_x, r)| \leq |\inf_x B_{C_{p_x}}(p_x, r)| = 0$$

for any $r > 0$

Convex bodies of bounded geometry

A convex body $C \subset \mathbb{R}^n$ is of *bounded geometry* if there exists $r_0 > 0$ such that

$$b(r_0) := \inf_{p \in C} |B_C(p, r_0)| > 0$$

Asymptotic cylinder

Let $C \subset \mathbb{R}^n$ be a convex body, $\{x_i\}_{i \in \mathbb{N}}$ a diverging sequence in C. Then $-x_i + C$ subconverges locally in Hausdorff distance to a convex cylinder K: an *asymptotic cylinder* of C

Asymptotic cylinders play a key role in both the characterization of sets of bounded geometry and in the proof of generalized existence of isoperimetric regions

It turns out also that the notion of bounded geometry is the right one to ensure the positivity of the isoperimetric profile.

Proposition

The following are equivalent

1. C is of bounded geometry
2. $I_C(v) > 0$, $\forall v > 0$
3. All asymptotic cylinders of C are convex bodies (have non-empty interior)

Moreover

- 1 is equivalent to $b(r) = \inf_{p \in C} |B_C(p, r)| > 0$ for *all* $r > 0$, and
- 2 is equivalent to $I_C(v_0) > 0$ for *some* $v_0 > 0$

Main issue for existence

The standard way to prove existence of an isoperimetric region of volume $v > 0$ is to take a minimizing sequence $E_i \subset C$ of sets with $|E_i| = v$ so that $P_C(E_i) \to I_C(v) = \inf\{P_C(F) : |F| = v\}$. Then one tries to extract a convergent subsequence

This fails in general in a non-compact space (since all or part of a minimizing sequence could escape off to infinity)

To treat this problem we need two main ingredients

1. density estimates
2. analyze the behavior of minimizing sequences

A related analysis was used by Leonardi and Rigot to obtain existence of isoperimetric regions in sub-Riemannian Carnot groups

Density estimates

Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry, $v_0 > 0$, and $\{E_i\}_{i\in\mathbb{N}} \subset C$ a sequence such that (for H half-space)

$$|E_i| \leq v_0 \ \forall i \in \mathbb{N}\,, \ \lim_{i\to\infty} |E_i| = v \in (0, v_0]\,, \ \liminf_{i\to\infty} P_C(E_i) \leq I_H(v)\,.$$

Take $m_0 \in (0, \frac{1}{2}]$ such that

$$m_0 < \min\left\{\frac{1}{2v_0}, \frac{\Lambda^n}{I_H(1)^n}\right\}.$$

Then there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset C$ such that

$$|E_i \cap B_C(x_i, 1)| \geq m_0|E_i| \qquad \text{for } i \text{ large enough.}$$

(Λ is a constant depending on the uniform geometry of C)

Behavior of minimizing sequences

Let $\{C_i\}_{i\in\mathbb{N}}$ a sequence of unbounded convex bodies converging locally in Hausdorff distance to an unbounded convex body C with $0 \in C$. Let $E_i \subset C_i$ be a sequence of measurable sets with volumes v_i converging to $v > 0$ and uniformly bounded perimeter. Assume $E \subset C$ is the $L^1_{loc}(\mathbb{R}^n)$ limit of $\{E_i\}_{i\in\mathbb{N}}$. Then, passing to a (non relabeled) subsequence, there exist diverging radii $r_i > 0$ such that

$$E_i^d := E_i \setminus B(0, r_i)$$

satisfies

(i) $|E| + \lim_{i\to\infty} |E_i^d| = v$.

(ii) $P_C(E) + \liminf_{i\to\infty} P_{C_i}(E_i^d) \leq \liminf_{i\to\infty} P_{C_i}(E_i)$.

Generalized isoperimetric regions

We say that a finite family $E^0, E^1, \ldots, E^k$ of sets of finite perimeter is a *generalized isoperimetric region* in C if $E^0 \subset C = K^0$, E^i is contained in an asymptotic cylinder K^i for $i \geq 1$ and, for any family of sets $F^0, F^1, \ldots, F^r$ such that $F^i \subset \tilde{K}^i$ and $\sum_{i=0}^k |E^i| = \sum_{i=0}^r |F^i|$, we have

$$\sum_{i=0}^{k} P_{K^i}(E^i) \leq \sum_{i=0}^{r} P_{\tilde{K}^i}(F^i).$$

Moreover, each E^i is an isoperimetric region in K^i with volume $|E^i|$.

Main existence result

Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry, $v_0 > 0$. There exists $\ell \in \mathbb{N}$ such that: for any minimizing sequence $\{F_i\}_i$ for volume v_0, one can find a (not relabeled) subsequence $\{F_i\}_i$ such that, for every $j \in \{0, \dots, \ell\}$, there exist

- a divergent sequence $\{x_i^j\}_i$, a sequence of sets $\{F_i^j\}_i$, an asymptotic cylinder $K^j \in \mathcal{K}(C)$, an isoperimetric region $E^j \subset K^j$ (possibly empty), ($x_i^0 = 0$, $K^0 = C$) such that

(i) $F_i^{j+1} \subset F_i^j \subset F_i$ for all $i \in \mathbb{N}$ and $j \in \{0, \dots, \ell - 1\}$;

(ii) $\lim_{i\to\infty} -x_i^j + C = K^j$ locally in Hausdorff distance;

(iii) $\lim_{i\to\infty} -x_i^j + F_i^j = E^j \subset K^j$ in $L^1_{loc}(\mathbb{R}^n)$;

(iv) for any $0 \le q \le \ell$, $E^0, E^1, \dots, E^q$ is a generalized isoperimetric region of volume $\sum_{j=0}^q |E^j|$.

(v) $\sum_{j=0}^\ell |E^j| = v_0$;

(vi) $I_C(v_0) = \sum_{j=0}^\ell I_{K^j}(|E^j|)$.

Sketch of the proof

Let E^0 be the L^1_{loc}-limit (possibly empty) of F_i. Find a diverging sequence r_i^0 such that $F_i^1 := F_i \setminus B(0, r_i^0)$ satisfies

1. $|E^0| + \lim_{i\to\infty} |F_i^1| = v_0$,
2. $P_C(E^0) + \liminf_{i\to\infty} P_C(F_i^1) \leq \liminf_{i\to\infty} P_C(F_i)$.

Then

1. If $|E^0| > 0$ then E^0 is isoperimetric for its volume
2. $P_C(E^0) + \liminf_{i\to\infty} P_C(F_i^1) = I_V(v_0)$

So we have

1. Either $|E^0| = v_0$ and E^0 is isoperimetric for volume v_0, or
2. $|E^0| < v_0$.

Sketch of the proof (continued)

Assume $|E^0| < v_0$. By the density estimate, one can find a sequence $x_i \in F_i^1$ such that

$$|F_i^1 \cap \overline{B}_C(x_i^1, 1)| \geq m_0 |F_i^1|$$

for i large. Since x_i^1 diverges,

1. $-x_i^1 + C$ subconverges to an asymptotic cylinder K^1.
2. $-x_i^1 + F_i^1$ subconverges L^1_{loc} to $E^1 \subset K^1$ of volume

$$v_0 - |E^0| \geq |E^1| \geq m_0(v_0 - |E^0|)$$

Sketch of the proof (induction)

Define F_i^2 finding a sequence of diverging radii r_i^2 by

$$-x_i^1 + F_i^2 = (-x_i^1 + F_i^1) \setminus B(0, r_i^2)$$

Then we have

1. $|E^1| + \lim_{i\to\infty} |F_i^2| = \lim_{i\to\infty} |F_i^1|$,
2. $P_{K^1}(E^1) + \liminf_{i\to\infty} P_C(F_i^2) \leq \liminf_{i\to\infty} P_C(F_i^1)$

Reasoning as above we find

1. $|E^0| + |E^1| + \lim_{i\to\infty} |F_i^2| = v_0$
2. $P_C(E^0) + P_{K^1}(E^1) + \liminf_{i\to\infty} P_C(x_i^1 + F_i^2) = I_C(v_0)$

and we use these formulas as a basis for an induction scheme

End of the proof (finiteness)

There exists a constant $\beta > 0$ (only depending on C and an upper bound of v_0) such that

$$|E^j| \geq \beta$$

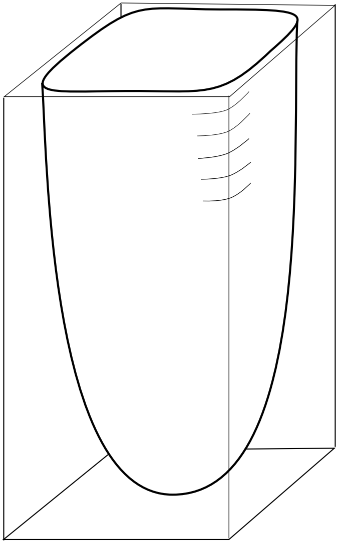
So we only have a finite number of steps

Remarks

1. A corresponding result was obtained by Nardulli for manifolds of controlled geometry
2. The generalized existence theorem allows us to work as if we had existence of isoperimetric regions.
3. It is still an open problem to decide whether there is existence of isoperimetric regions in a convex body of uniform geometry.

In the following slide we show an example

Conjectured example of non-existence of isoperimetric sets



Concavity of $I_C^{n/(n-1)}$ (sketch)

1. In the compact case the $\left(\frac{n}{n-1}\right)$-concavity of the profile is proved when the boundary is smooth by deforming an isoperimetric region. The general case follows by approximation in Hausdorff distance
2. In the bounded geometry case we have to approximate a convex set by sets with (bounded geometry and) smooth boundary whose asymptotic cylinders have also smooth boundary
3. A major consequence is the connectedness of generalized isoperimetric regions. In some cases this allow to prove existence of isoperimetric regions: for instance, for convex bodies of revolution

Small volumes (bounded convex bodies)

Given a bounded convex body, the aperture of tangent cones at the boundary is a lower semicontinuous function. This implies the existence of boundary points with the smallest possible aperture. All these cones have the same isoperimetric profile, that will be denoted by $I_{C_{\min}}$.

Theorem

Let $C \subset \mathbb{R}^n$ be a bounded convex body. Then

$$\lim_{v \to 0} \frac{I_C(v)}{I_{C_{\min}}(v)} = 0.$$

Moreover, if we scale a sequence of isoperimetric regions (E_i) whose volumes go to 0, we get Hausdorff convergence to a ball in a tangent cone with the smallest possible aperture.

Sketch of proof

$(E_i)_i$ isoperimetric regions with $|E_i| \to 0$, $x_i \in \partial C \cap E_i$. We scale E_i with center x_i and factor λ_i to get $\tilde{E}_i \subset \tilde{C}_i$ with volume 1. Passing to a subsequence (uniformly bounded diameter by density estimates and connectedness) we have $\tilde{E}_i \to E_0$ in $C_0 = -x_0 + C_{x_0}$, where x_0 is assumed to be the limit of (x_i). We have

$$I_{C_0}(1) \leq P_{C_0}(E_0) \leq \liminf_{i\to\infty} P_{\tilde{C}_i}(\tilde{E}_i) = \liminf_{i\to\infty} I_{\tilde{C}_i}(1),$$

and so

$$\liminf_{i\to\infty} \frac{I_C(v_i)}{I_{C_0}(v_i)} = \liminf_{i\to\infty} \frac{\lambda_i^{n-1} I_C(1/\lambda_i^n)}{\lambda_i^{n-1} I_{C_0}(1/\lambda_i^n)} = \liminf_{i\to\infty} \frac{I_{\tilde{C}_i}(1)}{I_{C_0}(1)} \geq 1.$$

On the other hand, $I_C \leq I_{C_0}$ for any tangent cone, so that

$$\limsup_{i\to\infty} \frac{I_C(v_i)}{I_{C_0}(v_i)} \leq 1.$$

We claim that $I_{C_0} = I_{C_{\min}}$: otherwise $I_{C_0}/I_{C_1} \geq \alpha > 1$ for some other tangent cone C_1 and some constant α. Since $I_C \leq I_{C_1}$ we should have

$$\frac{I_C}{I_{C_0}} = \frac{I_C}{I_{C_1}} \frac{I_{C_1}}{I_{C_0}} \leq \frac{1}{\alpha} < 1,$$

yielding a contradiction that proves the claim.

Comments

1. Generalized existence provides a similar proof in the unbounded case
2. When C is a bounded convex body with smooth boundary, isoperimetric regions concentrate near the maxima of the mean curvature of ∂C (convexity is not necessary) by the results of Fall

Open problems

1. Find a proof of existence or a counterexample to non-existence
2. Connectedness of isoperimetric regions in the $C^{2,\alpha}$ case and in the general case (using concavity). Connectedtedness of the complement should also work except for cylinders
3. Is there an alternative proof of concavity based on the bounded case (Milman)?
4. Regularity at the boundary of isoperimetric regions

Thanks for your attention

and Happy Birthday!!