ISOPERIMETRIC SETS IN CARNOT GROUPS WITH A SUB-FINSLER STRUCTURE

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In memory of Umberto Massari, an outstanding mathematician

ABSTRACT. The aim of this note is to provide a complete proof of existence of isoperimetric sets in sub-Finsler Carnot groups, and to establish some properties of such sets.

1. INTRODUCTION

The aim of this note is to provide a complete proof of existence of isoperimetric sets in Carnot groups endowed with a sub-Finsler structure, and to obtain some properties of such sets. In the particular case of a sub-Riemannian structure, such results were provided by Leonardi and Rigot [21]. In the first Heisenberg group $\mathbb{H}^1$ with a *symmetric* sub-Finsler structure a sketch of the proof of the existence result appeared in Franceschi et al. [12]. For nilpotent groups with a sub-Finsler structure existence of isoperimetric sets was proven by Pozuelo [30]. It is worth mentioning that the asymmetry of the sub-Finsler structure prevents the straightforward use of results known for sets of finite perimeter in metric spaces, since an asymmetric sub-Finsler structure does not have an associated distance as the symmetry property is missing.

We follow for the most part the strategy of Leonardi and Rigot, introducing in several parts of the proof some simplifying, from the conceptual point of view, ideas that have been developed in recent years and can be found in works by Morgan [26, 27], Ritoré and Rosales [32], Galli and Ritoré [14], Nardulli [28], Leonardi et al. [22], and Antonelli et al. [3, 6, 4]. In the Riemannian case one can follow §4.4 in the recent monograph [31].

An essential technique when working with isoperimetric sets (or variational problems with a volume constraint) is to have a geometric way of restoring the original volume of a set after a geometric operation has been performed on the set. For this purpose, flows of vector fields have been used in Euclidean spaces or Riemannian manifolds (e.g., §1.4.5 in [31] and the references therein), Euclidean dilations in [8], or Cheeger sets in [30].

The main differences of our proof with the one in Leonardi and Rigot's paper [21] are the geometric use of the strict concavity of the isoperimetric profile function and the use of structure results for minimizing sequences introduced in [32] to simplify the existence proof, and the systematic use of dilations to adjust the volume of a set after some previous geometric transformation. Although our result is contained in the one by Pozuelo on nilpotent groups with a sub-Finsler structure, the strict concavity of the isoperimetric profile greatly simplifies the proof and allows us to obtain the stronger result that an

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isoperimetric set in a Carnot group with a sub-Finsler structure is indecomposable, a notion of connectedness for finite perimeter sets.

In addition to the existence of isoperimetric sets we also prove some mild regularity results for their boundaries. In particular, we show in Theorem 5.2 that the boundary of an isoperimetric set is Ahlfors regular. Very few results are known about the regularity of isoperimetric boundaries in Carnot groups, or even in the simpler case of the first Heisenberg group $\mathbb{H}^1$. A lack of a regularity result similar to the Euclidean one proven by Gonzalez et al. [18] is one of the major drawbacks of the theory. Giovannardi and Ritoré [17] proved such a regularity for (X, Y)-Lipschitz surfaces in the first Heisenberg group $\mathbb{H}^1$ with prescribed mean curvature, a concept introduced by Massari in [23] and later considered in [19, 7]. In the paper [17] is proven that isoperimetric sets with (X, Y)-Lipschitz boundary have *constant* prescribed mean curvature.

Our main result in this paper is Theorem 2.16, where we prove

> *If $\mathbb{G}$ is a Carnot group endowed with a sub-Finsler norm $||\cdot||_K$ then K-isoperimetric sets exist on $\mathbb{G}$ for any positive volume. Moreover, the isoperimetric profile I_K is strictly concave and any isoperimetric set is essentially bounded and undecomposable.*

We have organized this paper into several sections. In the next one we present some preliminaries on Carnot groups, sub-Finsler norms and the sub-Finsler perimeter. In the third section we include some results on the isoperimetric profile of a sub-Finsler Carnot group. In the fourth one we complete the proof of existence of isoperimetric sets, where the concavity of the isoperimetric profile will play an important role. In the last section, we prove some density estimates for isoperimetric sets and the Ahlfors regularity of their boundaries.

2. Preliminaries

2.1. **Carnot groups.**

We refer the reader to the first chapter in Vittone's Ph.D. Thesis [34] for a good introduction to the geometry of Carnot groups.

A *stratified* or *Carnot group* is a connected and simply connected n-dimensional real Lie group $\mathbb{G}$ whose Lie algebra $\mathcal{G}$ admits a step-k stratification: a family of subspaces $V_1, \ldots, V_k \subset \mathcal{G}$ such that

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_k,$$

and

$$[V_1, V_j] = V_{j+1}. \qquad \text{for all } j = 1, \ldots, k, \text{ with } V_{k+1} = \{0\}. \tag{2.1}$$

The subspaces V_i are known as *strata* or *layers* and V_1, also denoted by $\mathcal{H}$, is usually referred to as the *first layer* or the *horizontal distribution*. Iterated Lie brackets of vector fields in V_1 generate the whole Lie algebra $\mathcal{G}$. A consequence of (2.1) is that

$$[V_i, V_j] \subseteq V_{i+j} \qquad \text{for all } i, j \in \mathbb{N} \text{ with } V_{i+j} = \{0\} \text{ if } i + j > k. \tag{2.2}$$

Hence, letting $\mathcal{G}_{i+1} = [\mathcal{G}, \mathcal{G}_i]$ for all $i \in \mathbb{N}$, with $\mathcal{G}_0 = \mathcal{G}$, we obtain that every element $\mathcal{G}_i$ of the lower central series satisfies

$$\mathcal{G}_i \subseteq V_{i+1} \oplus \cdots \oplus V_k, \qquad i = 0, \ldots, k,$$

and $\mathcal{G}_k = \{0\}$. This implies that $\mathbb{G}$ is a nilpotent group of step at most k. In particular, the exponential map $\exp : \mathcal{G} \to \mathbb{G}$ is a diffeomorphism, as proven in Proposition 1.2(a) in [11].

Choosing a basis of left-invariant vector fields $X_1, \dots, X_n$ of $\mathscr{G}$ provides an identification between $\mathbb{R}^n$ and $\mathscr{G}$. The composition of the inverse $\log = \exp^{-1} : \mathbb{G} \to \mathscr{G}$ with the identification provides a global chart on $\mathbb{G}$ whose coordinates are called *canonical coordinates of the first kind*. In such coordinates the group product can be recovered from the Baker-Campbell-Hausdorff-Dynkin formula, see formula (1.7.3) in [9]. Given $X, Y \in \mathscr{G}$, and letting $Z = \log(\exp X \cdot \exp Y)$ we get

$$
\begin{aligned}
Z = X + Y + \sum_{m=1}^{\infty} \frac{(-1)^m}{m+1} \cdot \Bigg\{ \sum_{\substack{a_1,\dots,a_m \geqslant 0,\\ b_1,\dots,b_m \geqslant 0,\\ a_i+b_i>0}} \frac{1}{1+\sum_{i=1}^m a_i} \times \\
\times \left(\frac{(\operatorname{ad} X)^{a_1}}{a_1!} \circ \frac{(\operatorname{ad} Y)^{b_1}}{b_1!} \circ \dots \circ \frac{(\operatorname{ad} X)^{a_m}}{a_m!} \circ \frac{(\operatorname{ad} Y)^{b_m}}{b_m!} \right)(X) \Bigg\},
\end{aligned} \tag{2.3}
$$

where $(\operatorname{ad} X)(Y) = [X, Y]$ for any $X, Y \in \mathscr{G}$. If $\mathbb{G}$ is nilpotent of step k then the sum (2.3) only extends from $m = 1$ to k and, moreover $\sum_{i=1}^m (a_i + b_i) \leqslant k$.

Assume that we have ordered the vector fields $X_1, \dots, X_n$ so that consecutively we have bases for $V_1, \dots, V_k$, and let

$$d = \dim V_1.$$

We want to express the group product $\cdot$ in canonical coordinates of the first kind. Taking $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, and $X = \sum_{i=1}^n x_i X_i$, $Y = \sum_{i=1}^n y_i X_i$, from (2.3) we obtain

$$x \cdot y = x + y + P(x, y), \tag{2.4}$$

where $P = (P_1, \dots, P_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial of degree at most k. As for any $X, Y \in \mathscr{G}$ the last summand in (2.3) belongs to $\mathscr{G}_1 \subseteq V_2 \oplus \dots \oplus V_k$ we have $P_1 = \dots = P_d = 0$. Moreover, if $V_j = \operatorname{span}\{X_{a+1}, \dots, X_{a+b}\}$ then the last summand in (2.3) belongs to V_j only if the involved vectors belong to $V_1 \oplus \dots \oplus V_{j-1}$. This means that for any $s \in \{a+1, \dots, a+b\}$, the polynomial P_s only depends on $x_1, y_1, \dots, x_a, y_a$. As $a < s$ this implies that

$$P_s(x, y) = P_s(x_1, \dots, x_{s-1}, y_1, \dots, y_{s-1}), \text{ for } s > d. \tag{2.5}$$

Because of (2.4) and (2.5), the differential of a left-traslation in canonical coordinates of the first kind is given by a lower triangular matrix with ones in the diagonal. Hence the Lebesgue measure in $\mathbb{R}^n$ with canonical coordinates is left-invariant and coincides (up to a constant) with the Haar measure of $\mathbb{G}$.

In addition, if we want to express any vector field $X_1, \dots, X_d$ in the first layer in terms of $\partial/\partial x_1, \dots, \partial/\partial x_n$, as $X_j(x)$ is the image by $(d\ell_x)_0$ of the j-th coordinate vector in $\mathbb{R}^n$, we would have

$$X_j(x) = \frac{\partial}{\partial x_j} + \sum_{i=d+1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i}, \quad j = 1, \dots, d, \tag{2.6}$$

where $a_{ij}(x) = \frac{\partial P_i}{\partial x_j}(x, 0) = a_{ij}(x_1, \dots, x_{i-1})$.

In a Carnot group $\mathscr{G}$ we can define in canonical coordinates a family of *intrinsic dilations* $h_\lambda : \mathbb{G} \to \mathbb{G}$, for $\lambda \in \mathbb{R}$, by the formula

$$h_\lambda(z_1, \dots, z_k) = (\lambda z_1, \dots, \lambda^i z_i, \dots, \lambda^k z_k), \tag{2.7}$$

where $(z_1, \dots, z_k) = (x_1, \dots, x_n)$ with $z_i \in \mathbb{R}^{\dim V_i}$. Each z_i correspond to the coordinates in $(x_1, \dots, x_n)$ associated to the vector fields in V_i. Because of (2.2) the intrinsic dilations satisfy

$$[h_\lambda(X), h_\lambda(Y)] = h_\lambda([X,Y]), \text{ for all } X, Y \in \mathcal{G}.$$

By (2.3) we get $h_\lambda(x \cdot y) = h_\lambda(x) \cdot h_\lambda(y)$ for all $x, y \in \mathbb{G}$. Sometimes we will write λE instead $h_\lambda(E)$ for subsets $E \subseteq \mathbb{G}$.

The *homogeneous dimension* of the group $\mathbb{G}$ is defined by

$$Q = \sum_{i=1}^{k} i \dim(V_i).$$

The Haar measure of a Borel measurable set $E \subset \mathbb{G}$ will be denoted by $|E|$. As the Haar measure coincides with the Lebesgue measure in canonical coordinates, the expression (2.7) immediately provides

$$|h_\lambda(E)| = \lambda^Q |E|,$$

for any measurable set $E \subseteq \mathbb{G}$ and any $\lambda > 0$. We also have that, for any left-translation ℓ_x with $x \in \mathbb{G}$, we have

$$|\ell_x(E)| = |E|$$

for any measurable set $E \subseteq \mathbb{G}$.

2.2. **Sub-Finsler structures in a Carnot group.** Our notion of norm is the one of *asymmetric* norm. This is a non-negative function $||\cdot|| : V \to \mathbb{R}$ defined on a finite-dimensional real vector space V satisfying

1. $||v|| = 0$ if and and only if $v = 0$,
2. $||\lambda v|| = \lambda ||v||$, for all $\lambda \geqslant 0$ and $v \in V$, and
3. $||v + w|| \leqslant ||v|| + ||w||$, for all $v, w \in V$.

We stress the fact that we are not assuming the standard symmetry property $||-v|| = ||v||$.

Associated to a given a norm $||\cdot||$ in V we have the set $K = \{u \in V : ||u|| \leqslant 1\}$, which is compact, convex and includes 0 in its interior. Reciprocally, given a compact convex set K with $0 \in \text{int}(K)$, the function $||u||_K = \inf\{\lambda \geqslant 0 : u \in \lambda K\}$ defines a norm in V so that $K = \{u \in V : ||u||_K \leqslant 1\}$.

Given a norm $||\cdot||$ and an scalar product $\langle \cdot, \cdot \rangle$ in V, we consider the dual norm $||\cdot||_*$ of $||\cdot||$ with respect to $\langle \cdot, \cdot \rangle$ defined by

$$||u||_* = \sup_{||v|| \leqslant 1} \langle u, v \rangle.$$

The dual norm is the support function h of the unit ball $K = \{u \in V : ||u|| \leqslant 1\}$ with respect to the scalar product $\langle \cdot, \cdot \rangle$.

A norm is said to be *smooth* if it is C^∞ in $V \setminus \{0\}$. It is *strictly convex* if

$$||\lambda u + (1-\lambda) v|| < 1, \quad \text{for all } \lambda \in (0,1), \text{when } u \neq v, ||u|| = ||v|| = 1.$$

Given $u \in V$, the compactness of the unit ball of $||\cdot||$ and the continuity of $||\cdot||$ implies the existence of $u_0 \in V$ satisfying equality $||u||_* = \langle u, u_0 \rangle$. Moreover, it can be easily checked that $||u_0|| = 1$. In general, a point u_0 satisfying this property is not unique, but uniqueness follows from the assumption that $||\cdot||$ is strictly convex: this is proved by contradiction assuming the existence of another point u_0' with $||u_0'|| \leqslant 1$ satisfying $||u||_* = \langle u, u_0' \rangle$. Of course u_0' must also satisfy $||u_0'|| = 1$. Then all the points v in the segment $[u_0, u_0']$ satisfy $||v|| \leqslant 1$ and $||u||_* = \langle u, v \rangle$; hence $||v|| = 1$. But this contradicts the strict convexity of $||\cdot||$ unless $u_0 = u_0'$. We shall define $\pi(u)$ as the only vector satisfying $||\pi(u)|| = 1$ and

$$h(u) = ||u||_* = \langle u, \pi(u) \rangle.$$

If $\lambda > 0$ then it is easily checked that $\pi(\lambda u) = \pi(u)$.

We say that a convex body K is of class C^ℓ_+, with $\ell \geqslant 2$ when ∂K is of class C^ℓ, $\ell \geqslant 2$, and the principal curvatures of ∂K are everywhere positive. Hence the Gauss map $N : \partial K \to \mathbb{S}^1$ to the unit sphere is a diffeomorphism of class $C^{\ell-1}$. Since $\pi = N^{-1}$ we conclude that π is of class $C^{\ell-1}$. Moreover, by Corollary 1.7.3 in [33] we have

$$\nabla h(u) = N^{-1}\Big(\frac{u}{|u|}\Big),$$

and so h is of class C^ℓ. If K is of class C^2_+ then it is strictly convex and, if $0 \in \operatorname{int}(K)$, then the norm $||\cdot||_K$ is strictly convex.

We now consider left-invariant norms in Carnot groups. Let $\mathbb{G}$ be a Carnot group and let $\mathcal{H}$ be the *horizontal distribution* in $\mathbb{G}$, determined by the left-invariant vector fields of the first layer. For $p \in \mathbb{G}$, we denote by $\mathcal{H}_p \subset T_p\mathbb{G}$ the subspace $\{X_p : X \in \mathcal{H}\}$. Given a norm $||\cdot||_0$ in $\mathcal{H}_0$, we extend it by left-invariance to a norm $||\cdot||$ in the whole horizontal distribution $\mathcal{H}$ by means of the formula

$$||v||_p = ||d\ell_p^{-1}(v)||_0, \qquad p \in \mathbb{G}, v \in \mathcal{H}_p. \tag{2.8}$$

In particular, for a *horizontal* vector field $f_1X_1 + \ldots f_dX_d$, its norm at a point $p \in \mathbb{G}$ is given by $||f_1(p)(X_1)_0 + \cdots + f_d(p)(X_d)_0||$. A *sub-Finsler norm* in $\mathbb{G}$ is a left-invariant norm in $\mathcal{H}$. If $K \subset \mathcal{H}_0$ is the closed unit ball for $||\cdot||$, we will frequently denote the sub-Finsler norm by $||\cdot||_K$.

We consider the norm $(||\cdot||_0)_*$, dual to $||\cdot||_0$ in $\mathcal{H}_0$, and we extend it by left-invariance to a norm $||\cdot||_*$ in $\mathcal{H}$. It can be checked that $(||\cdot||_*)_p$ is the dual norm to $||\cdot||_p$ since

$$\begin{aligned}(||v||_*)_p = (||d\ell_p^{-1}(v)||_0)_* &= \sup ||w||_0 \leqslant 1, w \in \mathcal{H}_0 \langle d\ell_p^{-1}(v), w\rangle \\ &= \sup ||w'||_p \leqslant 1, w' \in \mathcal{H}_p\langle v, w'\rangle \\ &= (||v||_p)_*.\end{aligned}$$

When $||\cdot||_0$ is C^l_+ with $l \geqslant 2$, all norms $||\cdot||_p$ are C^l_+. Given a horizontal vector field U of class C^1, we define $\pi(U)$ as the C^1 horizontal vector field satisfying

$$||U||_* = \langle U, \pi(U)\rangle, \tag{2.9}$$

or, equivalently, $(||U_p||_p)_* = \langle U_p, \pi(U)_p\rangle$ for all $p \in \mathbb{G}$. We recall that $\pi(fU) = \pi(U)$ for any positive smooth function f.

A curve $\gamma : [a,b] \subset \mathbb{R} \to \mathbb{G}$ of class C^1 is *horizontal* if $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for all $t \in [a,b]$. Given a sub-Finsler norm, the *sub-Finsler length* of a curve $\gamma : [a,b] \to \mathbb{G}$ is defined by

$$L(\gamma) = \int_a^b ||\gamma'(t)||\, dt.$$

Since any two given points in $\mathbb{G}$ can be connected by a smooth horizontal curve by Chow's Theorem (Theorem 2.1.2 in [25]), we may define the *sub-Finsler distance* in $\mathbb{G}$ by

$$d(p,q) = \inf L(\gamma),$$

where $\gamma : [a,b] \to \mathbb{G}$ is any C^1 horizontal curve with $\gamma(a) = p$, $\gamma(b) = q$. Observe that d is an *asymmetric distance* satisfying $d(p,q) \geqslant 0$, $d(p,q) = 0$ if and only if $p = q$, and the triangle inequality, but not the symmetry property $d(p,q) = d(q,p)$ because $||\cdot||$ is not symmetric. Observe also that the sub-Finsler distance is invariant by left-translations and that

$$d(h_\lambda(p), h_\lambda(q)) = \lambda\, d(p,q)$$

for any $\lambda > 0$. This follows from the fact that $h_\lambda \circ \gamma$ is horizontal when γ is horizontal and $||(h_\lambda \circ \gamma)'(t)|| = \lambda\, ||\gamma'(t)||$ for all $t \in [a,b]$.

The *sub-Finsler open ball* of center $p \in \mathbb{G}$ and radius $r > 0$ is defined by

$$B_K(p,r) = \{q \in \mathbb{G} : d(p,q) < r\}.$$

For a left-translation ℓ_q we have $\ell_q(B(p,r)) = B(\ell_q(p), r)$ and, for dilations, $h_\lambda(B(p,r)) = B(h_\lambda(p), \lambda r)$.

Given two convex bodies $K, K' \subset V_1$ containing 0 in its interior, we can compare the sub-Finsler distances by observing that there exist constants $\alpha, \beta > 0$ such that

$$\alpha\, ||x||_{K'} \leqslant ||x||_K \leqslant \beta\, ||x||_{K'}, \quad \text{for all } x \in V_1. \tag{2.10}$$

If $\gamma : [a,b] \to \mathbb{G}$ is a C^1 curve connecting the points $p, q \in \mathbb{G}$ then we have

$$\alpha\, L_{K'}(\gamma) \leqslant L_K(\gamma) \leqslant \beta\, L_{K'}(\gamma).$$

Henceforth

$$\alpha\, d_{K'}(p,q) \leqslant d_K(p,q) \leqslant \beta\, d_{K'}(p,q)$$

for all $p, q \in \mathbb{G}$.

2.3. **Convolutions in Carnot groups.** For this section we refer the reader to §1.2.7 in Vittone's Ph.D. Thesis [34] or Proposition 1.20 and pages 21–22 in Folland-Stein [11]. We start with a function $\varphi \in C_0^\infty(\mathbb{G})$ satisfying

$$0 \leqslant \varphi \leqslant 1, \quad \int_{\mathbb{G}} \varphi \, d\mathbb{G} = 1, \quad \varphi(x^{-1}) = \varphi(x).$$

For $\varepsilon > 0$ we define

$$\varphi_\varepsilon(x) = \varepsilon^{-Q} \varphi(h_{1/\delta}(x)), \quad x \in \mathbb{G},$$

and, for a function $f \in L^1_{loc}(\mathbb{G})$,

$$f^\varepsilon(x) = (\varphi_\varepsilon \star f)(x) = \int_{\mathbb{G}} \varphi_\varepsilon(y) f(y^{-1} \cdot x)\, d\mathbb{G}(y) = \int_{\mathbb{G}} \varphi_\varepsilon(x \cdot y^{-1}) f(y)\, d\mathbb{G}(y).$$

Given a function φ in the previous conditions, we define, for every set $A \subset \mathbb{G}$ and $\varepsilon > 0$, the set A_ε as $h_{1/\varepsilon}(\mathrm{supp}(\varphi)) \cdot A$, where supp is the standard support of a function and, for any pair of subsets $B, C \subseteq \mathbb{G}$, its Minkowski product $B \cdot C$ is defined as $\{b \cdot c : b \in B, c \in C\}$, where $b \cdot c$ is the product in $\mathbb{G}$. We have the following properties

Proposition 2.1. *Let $\Omega \subseteq \mathbb{G}$ be an open set.*

1. $\mathrm{supp}(f^\varepsilon) \subset \mathrm{supp}(f)_\varepsilon$.
2. *If $f \in L^1_{loc}(\Omega)$ then $f^\varepsilon \in C^\infty(\Omega_\varepsilon)$.*
3. *If $f \in C^0(\Omega)$ then f^ε converges to f uniformly on compact subsets of Ω.*
4. *If $f \in L^p_{loc}(\Omega)$ for $1 \leqslant p < \infty$ then $f^\varepsilon \to f$ in $L^p_{loc}(\Omega)$.*
5. *For every $f \in L^1(\Omega)$ and $g \in L^\infty(\Omega)$ we have*
$$\int_{\mathbb{G}} f^\varepsilon g \, d\mathbb{G} = \int_{\mathbb{G}} f g^\varepsilon \, d\mathbb{G}.$$
6. *$Xf^\varepsilon = (Xf)^\varepsilon$ for any $f \in C^1(\Omega)$ and each left-invariant vector field $X \in \mathcal{G}$.*

7. *If* $f \in \mathrm{Lip}_{loc}(\Omega, ||\cdot||_D)$ *then* $Xf^\varepsilon = (Xf)^\varepsilon$ *for almost everywhere* $x \in \Omega_\varepsilon$ *and*
$$\int_{\Omega'} ||-\nabla_h f^\varepsilon||_{K,*}\, d\mathbb{G} \to \int_{\Omega'} ||-\nabla_h f||_{K,*}\, d\mathbb{G}$$
on any bounded open set $\Omega' \subseteq \Omega$.

Proof. The proof follows, for the most part, the classical Euclidean one which can be found, for instance, in Theorem 1, page 123, of Evans-Gariepy [10].

Let us check the proof of 6, where the geometry of the Carnot group intervenes. Take $z \in \mathbb{G}$. We denote, as customary in the theory of smooth manifolds, the derivative of the smooth function g in the direction of X at the point z by $X_z g$ or $(Xg)(z)$. Then
$$X_z f^\varepsilon = \int_{\mathbb{G}} \varphi_\varepsilon(y)\, X_z h_y\, d\mathbb{G}(y),$$
where $h_y(x) = f(y^{-1}\cdot x) = (f \circ \ell_{y^{-1}})(x)$. As X is a left-invariant vector field, we have $X_z h_y = X_z(f \circ \ell_{y^{-1}}) = [(d\ell_{y^{-1}})_z X_z] f = X_{y^{-1}\cdot z} f$, and so the last integral is equal to
$$\int_{\mathbb{G}} \varphi_\varepsilon(y)(Xf)(y^{-1}\cdot z)\, d\mathbb{G}(y) = (Xf)^\varepsilon(z),$$
as stated.

To check 7, we take a D-orthonormal basis $X_1, \dots, X_d$ of the first layer. Then $X_i f$ exists almost everywhere for all $1 \leqslant i \leqslant d$ by Pansu-Rademacher Theorem [29]. By the same argument as in item 6, we have $Xf^\varepsilon = (Xf)^\varepsilon$ almost everywhere for any $\varepsilon > 0$, and so
$$\begin{aligned}\int_{\Omega'} \big|||-\nabla_h f^\varepsilon||_{K,*} - ||-\nabla_h f||_{K,*}\big|\, d\mathbb{G} &\leqslant C \int_{\Omega'} ||\nabla_h f^\varepsilon - \nabla_h f||_D\, d\mathbb{G} \\ &\leqslant C m^{1/2} \max_{1\leqslant i\leqslant m} \left\{ \int_{\Omega'} |X_i f^\varepsilon - X_i f|\, d\mathbb{G} \right\},\end{aligned}$$
where C is the constant in the inequality $||\cdot||_{K,*} \leqslant C||\cdot||_D$. By 4, $X_i f^\varepsilon$ converges to $X_i f$ in $L^1_{loc}(\Omega')$ for all $1 \leqslant 1 \leqslant d$. □

2.4. **The sub-Finsler perimeter in a Carnot group.** Let $E \subseteq \mathbb{G}$ be a measurable set, $||\cdot||_K$ the left-invariant norm associated to a convex body $K \subset V_1$ so that $0 \in \mathrm{int}(K)$, and $\Omega \subseteq \mathbb{G}$ an open subset. Let us fix on $\mathbb{G}$ a left-invariant Riemannian metric and let div and $d\mathbb{G}$ be the divergence and the left-invariant Riemannian volume with respect to this Riemannian metric. We say that E has locally finite K-perimeter in Ω if for any relatively compact open set $\Omega' \subseteq \Omega$, the quantity

$$P_K(E,\Omega') = \sup\left\{ \int_E \mathrm{div}(U)\, d\mathbb{G} \,:\, U \in \mathfrak{X}^1_{0,H}(\Omega'), ||U||_{K,\infty} \leqslant 1 \right\}. \tag{2.11}$$

called the K-perimeter of E in Ω', is finite. The quantity $P_K(E,\Omega')$ is called the (relative) perimeter of E in Ω'. In this expression, $\mathfrak{X}^1_{0,H}(\Omega')$ is the space of horizontal vector fields of class C^1 with compact support in Ω', and $||U||_{K,\infty} = \sup_{p\in\Omega'} ||U_p||_K$. The set E has finite K-perimeter in an arbitrary open set Ω' if (2.11) is finite.

The definition of perimeter is a particular case of the notion of function of bounded variation. Given an open subset $\Omega \subseteq \mathbb{G}$, we say that a function $f \in L^1_{loc}(\Omega)$ is of *bounded local variation* in Ω if for all relatively compact open subsets $\Omega' \subseteq \Omega$ the quantity

$$\mathrm{var}_K(f,\Omega') = \sup\left\{ \int_\Omega f\, \mathrm{div}(U)\, d\mathbb{G} \,:\, U \in \mathfrak{X}^1_{0,H}(\Omega'), ||U||_{K,\infty} \leqslant 1 \right\}, \tag{2.12}$$

called the *total variation* of f in Ω', is finite. If $E \subseteq \mathbb{G}$ is measurable then the characteristic function χ_E belongs to $L^1_{loc}(\mathbb{G})$ and we have

$$P_K(E, \Omega') = \operatorname{var}_K(\chi_E, \Omega'),$$

for any bounded open set Ω'. The function $f \in L^1(\Omega')$ has K-bounded variation in Ω' if (2.12) is finite.

For any convex body K and $\phi \in C^\infty(\Omega)$, we consider a left-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$. From the definition of the dual norm $|| \cdot ||_{K,*}$, for any relatively compact open set $\Omega' \subseteq \Omega$ and $U \in \mathcal{H}^1_0(\Omega')$ with $||U||_{K,\infty} \leqslant 1$, we obtain $\langle -\nabla_h \phi, U \rangle \leqslant || -\nabla_h \phi||_{K,*}$, and so

$$\operatorname{var}_K(\phi, \Omega') \leqslant \int_{\Omega'} \phi \operatorname{div}(U)\, d\mathbb{G} = \int_{\Omega'} \langle -\nabla_h \phi, U \rangle \, d\mathbb{G} \leqslant \int_{\Omega'} || -\nabla_h \phi||_{K,*} \, d\mathbb{G}. \tag{2.13}$$

If K is of class C^2_+ we can work with the C^1 vector field $U = \pi_K(-\nabla_h \phi)$ as in the classical Euclidean case, approximating the vector field by smooth vector fields when $\pi_K(-\nabla_h \phi) \neq 0$, to conclude

$$\operatorname{var}_K(\phi, \Omega') = \int_{\Omega'} || -\nabla_h \phi||_{K,*} \, d\mathbb{G}. \tag{2.14}$$

In the general case, when K is an arbitrary convex set, we approximate it in Hausdorff distance by a sequence of convex sets $\{K_i\}_{i \in \mathbb{N}}$ of class C^2_+ (e.g., Theorem 2.7.1 in [33]). For any vector field $U \neq 0$ with compact support we have

$$\begin{aligned} \int_{\Omega'} || -\nabla_h \phi||_{K,*} \, d\mathbb{G} &= \lim_{i \to \infty} \int_{\Omega'} || -\nabla_h \phi||_{K_i,*} \, d\mathbb{G} \\ &= \lim_{i \to \infty} \operatorname{var}_{K_i}(\phi, \Omega') \\ &\leqslant \lim_{i \to \infty} \int_{\Omega'} \phi \operatorname{div}\Big(\frac{U}{||U||_{K_i,\infty}}\Big)\, d\mathbb{G} \\ &\leqslant \operatorname{var}_K(\phi, \Omega'). \end{aligned}$$

This inequality, together with (2.13), implies the validity of the formula (2.14) for any smooth function $\phi \in C^\infty(\Omega)$, a relatively compact open subset $\Omega' \subseteq \Omega$, and an arbitrary convex body K with $0 \in \operatorname{int}(K)$. So we have proven

Proposition 2.2. *Let $(\mathbb{G}, || \cdot ||_K)$ be a Carnot group with a sub-Finsler norm, and $\Omega \subseteq \mathbb{G}$ an open set. Then the total variation of a function $\phi \in C^\infty(\Omega)$ is any relatively compact open set $\Omega' \subseteq \Omega$ is given by*

$$\operatorname{var}_K(\phi, \Omega') = \int_{\Omega'} || -\nabla_h \phi||_{K,*} \, d\mathbb{G}. \tag{2.15}$$

Remark 2.3. If $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathcal{H}$ with unit ball D then we have a sub-Riemannian structure on $\mathbb{G}$. Assuming that ϕ is a D-Lipschitz function, then $\nabla_h \phi$ exists almost everywhere in $\mathbb{G}$ by Pansu-Rademacher's Theorem [29] and formula (2.15) still holds for ϕ.

It is not difficult to prove that $B_K(x, r)$ has finite K-perimeter and to estimate it from above. Let U be a C^1 horizontal vector field in $\mathbb{G}$ with compact support satisfying $||U||_{K,\infty} \leqslant 1$, and $\{\varphi_t\}_{t \in \mathbb{R}}$ the associated one-parameter group of diffeomorphisms.

For fixed $z \in \mathbb{G}$, let $\gamma_z(t) = \varphi_t(z)$, $t \in \mathbb{R}$, be the integral curve of U with initial condition $\gamma_z(0) = z$. If $z \in B_K(x,r)$ then

$$d_K(z, \varphi_t(z)) \leqslant \int_0^t ||\gamma_z'(t)||_K \, dt = \int_0^t ||U_{\gamma_z(t)}||_K \, dt \leqslant t,$$

and so

$$d_K(x, \varphi_t(z)) \leqslant d_K(x,z) + d_K(z, \varphi_t(z)) \leqslant r + t.$$

This implies $\varphi_t(B_K(x,r)) \subseteq B_K(x, r+t)$ for all $t \in \mathbb{R}$. Hence, letting

$$\omega_K = |B_K(0,1)|,$$

we have, by the first variation of volume, Theorem 1.11 in [31],

$$\begin{aligned}\int_{B_K(x,r)} \operatorname{div}(U)\, d\mathbb{G} &= \frac{d}{dt}\Big|_{t=0} |\varphi_t(B_K(x,r+t))| \\ &= \lim_{t\to 0} \frac{|\varphi_t(B_K(x,t))| - |B_K(x,r)|}{t} \\ &\leqslant \lim_{t\to 0} \frac{|B_K(x,r+t)| - |B_K(x,r)|}{t} \\ &= \omega_K \lim_{t\to 0} \frac{(r+t)^Q - r^Q}{t} = Q\omega_K\, r^{Q-1}.\end{aligned}$$

Taking the supremum over all vector fields U with compact support in $\mathbb{G}$ and $||U||_{K,\infty} \leqslant 1$ we have

$$P_K(B_K(x,r)) \leqslant Q\omega_K\, r^{Q-1} \tag{2.16}$$

for all $x \in \mathbb{G}$ and $r > 0$.

The following properties of the perimeter are quite standard. We assume Ω to be a bounded open set, and E, F measurable sets.

1. $P_K(E,\Omega) = P_K(F,\Omega)$ whenever $E\Delta F = (E \cup F) \setminus (E \cap F)$ has measure 0.
2. $P_K(E \cup F, \Omega) + P_K(E \cap F, \Omega) \leqslant P_K(E,\Omega) + P_K(F,\Omega)$,
3. The function $E \mapsto P_K(E,\Omega)$ is lower semicontinuous with respect to the $L^1_{loc}(\Omega)$ topology.
4. The set function $\Omega \to P_K(E,\Omega)$ is the restriction to the open subsets of a finite Borel measure $P_K(E,\cdot)$ defined by
$$P_K(E,A) = \inf\big\{P_K(E,\Omega) : A \subset \Omega, \Omega \text{ open}\big\}.$$
5. If D is the unit ball of a sub-Riemannian structure on $\mathbb{G}$, $x \in \mathbb{G}$ and $\rho > 0$, then

$$\begin{aligned} &P_K(E \cap B_D(x,\rho)) \leqslant P_K(E, B_D(x,\rho)) + P_K(E \cap B_D(x,\rho), \partial B_D(x,\rho)),\\ &P_K(E \setminus B_D(x,\rho)) \leqslant P_K(E, \mathbb{G} \setminus \overline{B}_D(x,\rho)) + P_K(E \setminus B_D(x,\rho), \partial B_D(x,\rho)).\end{aligned} \tag{2.17}$$

Remark 2.4. We observe that the classical property $P_K(E,\Omega) = P_K(\mathbb{G}\setminus E, \Omega)$ does not hold here since the norm $||\cdot||_K$ is not symmetric. If the boundary of E is a C^1 hypersurface, then there holds

$$P_K(E,\Omega) = \int_{\partial E\cap\Omega} ||N_h||_{K,*}\, dS,$$

where N_h is the orthogonal projection to $\mathscr{H}$ of *inner* unit normal N to ∂E and dS is the Riemannian area element in ∂E, both with respect to a fixed left-invariant Riemannian metric on $\mathbb{G}$. This formula is proven in two steps, first considering the case when K is of class C^2_+ extending the vector field $\pi_K(N_h)$, and then approximating a general convex set by sets of class C^2_+. If $||\cdot||_K$ is asymmetric then so it is $||\cdot||_{K,*}$ and hence $P_K(E,\Omega) \neq P_K(\mathbb{G}\setminus E,\Omega)$ in general.

Theorem 2.5 (Approximations by smooth functions)**.** *Let $\Omega\subset\mathbb{G}$ an open set. Then for any function $u\in BV_K(\Omega)$ of bounded variation in Ω, there exists a sequence $\{u_i\}_{i\in\mathbb{N}}$ in $C^\infty(\Omega)$ such that*

1. $\lim_{i\to\infty} ||u_i-u||_{L^1(\Omega)} = 0$,
2. $\lim_{i\to\infty} \operatorname{var}_K(u_i,\Omega) = \operatorname{var}_K(u,\Omega)$.

Proof. As in Theorem 1.14 in [16]. One can also use the classical proof from Theorem 3.9 in [2] together with the group convolution. □

Remark 2.6. Should we consider a different left-invariant Riemannian metric on $\mathbb{G}$ we would obtain the same value for the perimeter up to some constant independent of the sets. We simply observe that $d\mathbb{G}$ is, up to a constant, the Haar measure of $\mathbb{G}$, and that the integral

$$\int_E \operatorname{div}(U)\, d\mathbb{G}$$

is the derivative at $s=0$ of the Haar measure of $\varphi_s(E)$, where $\{\varphi_s\}_{s\in\mathbb{R}}$ is the one-parameter group of diffeomorphisms associated to the vector field U (e.g., Theorem 1.11 in [31]).

Remark 2.7. Let us take a sub-Riemannian metric on V_1. We compute $\operatorname{div}(U)$ explicitly. Choose an orthonormal basis $X_1,\ldots,X_d$ in V_1 and extend it to an orthonormal basis $X_1,\ldots,X_n$ in $\mathcal{G}$ adapted to the layers. Let ∇ be the Levi-Civita connection of the Riemannian metric. A C^1 horizontal vector field U can be expressed as

$$U=\sum_{i=1}^d f_i X_i,$$

for some C^1 functions f_i, and so

$$\operatorname{div}(U)=\operatorname{div}\Big(\sum_{i=1}^d f_i X_i\Big)=\sum_{i=1}^d X_i f_i + f_i \operatorname{div}(X_i).$$

For $i=1,\ldots,n$ we have

$$\operatorname{div}(X_i)=\sum_{j=1}^n\langle\nabla_{X_j}X_i,X_j\rangle=\sum_{j=1}^n\langle\nabla_{X_i}X_j,Xj\rangle+\sum_{j=1}^n\langle[X_j,X_i],X_j\rangle=0,$$

since $\langle\nabla_{X_i}X_j,X_j\rangle=0$ and, in case $X_i\in V_s$, $X_j\in V_r$, the vector fields X_j and $[X_j,X_i]$ lie in the different strata V_r, V_{s+r} and they are orthogonal. So we have

$$\int_E \operatorname{div}(U)\, d\mathbb{G}=\int_E\bigg(\sum_{i=1}^d X_i f_i\bigg)\, d\mathbb{G}.$$

This expression implies that our definition of perimeter P_D, with D the closed unit ball in $\mathcal{H}$ associated to the sub-Riemannian metric, coincides with the classical sub-Riemannian perimeter in the Heisenberg group considered in [13].

If $E\subset\mathbb{G}$ is a set of locally finite K-perimeter in Ω then, for any relatively compact subset $\Omega'\subseteq\Omega$ and any vector field $U\in\mathfrak{X}^1_{0,H}(h_\lambda(\Omega'))$ we have

$$\int_{h_\lambda(\Omega')}\operatorname{div}(U)\, d\mathbb{G}=\lambda^Q\int_{\Omega'}(\operatorname{div}(U)\circ h_\lambda)\, d\mathbb{G}.$$

If $U = \sum_{i=1}^{d} f_i X_i$, with $X_1, \dots, X_d$ orthonormal for some scalar product on $\mathscr{H}$ then

$$\operatorname{div}(U) = \sum_{i=1}^{d} X_i f_i.$$

So we have

$$\operatorname{div}(U) \circ h_\lambda = \lambda^{-1} \operatorname{div}\Big(\sum_{i=1}^{d} (f_i \circ h_\lambda) X_i\Big).$$

Letting $U_\lambda = \sum_{i=1}^{d} (f_i \circ h_\lambda) X_i$ we get

$$\int_{h_\lambda(\Omega')} \operatorname{div}(U)\, d\mathbb{G} = \lambda^{Q-1} \int_{\Omega'} \operatorname{div}(U_\lambda)\, d\mathbb{G}.$$

Since $||U||_{K,\infty} = ||U_\lambda||_{K,\infty}$ we finally obtain

$$P_K(h_\lambda(E), h_\lambda(\Omega')) = \lambda^{Q-1} P_K(E, \Omega'). \tag{2.18}$$

Remark 2.8. When K is a centrally symmetric convex body (e.g., $K = -K$) containing 0 in its interior, $|| \cdot ||_K$ induces a truly distance with the symmetry property. In such case, the arguments in §5.3 of Miranda [24] imply that the perimeter in the associated metric space coincides with the K-perimeter defined here.

Remark 2.9. The sub-Riemannian perimeter for systems of vector fields satisfying the Hörmander condition was introduced by Garofalo and Nhieu in their remarkable paper [16]. Let us recall their definition. One considers in $\mathbb{R}^n$ a system $X = \{X_1, \dots, X_d\}$ of vector fields

$$X_j = \sum_{i=1}^{n} b_{ji} \frac{\partial}{\partial x_i}, \qquad j = 1, \dots, d,$$

with locally Lipschitz continuous coefficients b_{ji} satisfying Hörmander condition (e.g., Lie brackets of $X_1, \dots, X_d$ generate the tangent space to $\mathbb{R}^n$). Let

$$X_j^* = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_{ji}\, \cdot), \qquad j = 1, \dots, d,$$

denote the formal adjoint of X_j. Then the X-perimeter of a measurable set $E \subset \mathbb{R}^m$ inside an open set $\Omega \subseteq \mathbb{R}^n$ is defined as

$$P_X(E, \Omega) = \sup\Big\{ \int_E \Big(\sum_{j=1}^{d} X_j^* \phi_j \Big) d\mathscr{L}^m \,:\, \phi \in \mathscr{F}(\Omega) \Big\},$$

where $\mathscr{F}(\Omega) = \big\{\phi = (\phi_1, \dots, \phi_d) \in C_0^1(\Omega, \mathbb{R}^d) \,:\, ||\phi||_\infty = \sup_{x \in \Omega} \sum_{j=1}^{d} \phi_j^2 \leqslant 1\big\}$.

Let us consider now a Carnot group $\mathbb{G}$ with a Riemannian metric in the d-dimensional horizontal distribution $\mathscr{H}$, which we extend to a Riemannian metric in $\mathbb{G}$ so that the layers are orthogonal. We take an orthonormal basis $X_1, \dots, X_n$ of $\mathscr{G}$ adapted to the layers. We can express the vector fields $X_1, \dots, X_d$ in canonical coordinates of the first kind using formula (2.6), which immediately implies

$$X_j^* = -X_j, \qquad j = 1, \dots, d.$$

Hence the X-perimeter P_X and the sub-Riemannian perimeter P_D, associated to the closed unit ball $D \subset \mathscr{H}$, coincide.

Let K, K' bounded convex bodies containing 0 in its interior. Let $E \subset \mathbb{G}$ be a measurable set, $\Omega \subset \mathbb{G}$ an open set. Take $U \in \mathfrak{X}^1_{0,H}(\Omega)$ a horizontal vector field with $||U||_{K,\infty} \leqslant 1$. Hence $||\alpha U||_{K',\infty} \leqslant ||U||_{K,\infty} \leqslant 1$ by (2.10) and

$$\int_E \operatorname{div}(U)\, d\mathbb{G} = \alpha^{-1} \int_E \operatorname{div}(\alpha U)\, d\mathbb{G} \leqslant \alpha^{-1} P_{K'}(E, \Omega),$$

Taking supremum over the set $\mathfrak{X}^1_{0,H}(\Omega)$ of C^1 horizontal vector fields with compact support in Ω and $||\cdot||_K \leqslant 1$, we get $P_K(E,\Omega) \leqslant \alpha^{-1} P_{K'}(E,\Omega)$. In a similar way we get the inequality $\beta^{-1} P_{K'}(E,\Omega) \leqslant P_K(E,\Omega)$, so that we have

$$\beta^{-1} P_{K'}(E,\Omega) \leqslant P_K(E,\Omega) \leqslant \alpha^{-1} P_{K'}(E,\Omega). \tag{2.19}$$

As a consequence, E has locally finite K-perimeter if and only if it has locally finite K'-perimeter.

Hence, fixing a sub-Riemannian metric on $\mathcal{H}$ with unit ball D, there follows the existence of a constant $C_K > 0$ such that

$$C_K^{-1} P_D \leqslant P_K \leqslant C_K P_D. \tag{2.20}$$

As a consequence of the previous discussions, we now prove three important results, a compactness theorem for sets of finite K-perimeter, a version of Ambrosio's localization lemma, and a version of the local isoperimetric inequality.

Theorem 2.10 (Compactness). *Let $\Omega \subseteq \mathbb{G}$ be an open set in a Carnot group $\mathbb{G}$ equipped with a sub-Finsler norm $||\cdot||_K$, and $\{E_i\}_{i\in\mathbb{N}}$ a sequence of sets of uniformly bounded volume and uniformly bounded K-perimeter $P_K(E_i,\Omega')$ in any relatively compact subset $\Omega_i' \subseteq \Omega$. Then there is a set of locally finite K-perimeter $E \subset \mathbb{G}$ and a subsequence of $\{E_i\}_{i\in\mathbb{N}}$ converging in $L^1_{loc}(\Omega)$ to E.*

Proof. We consider a sub-Riemannian metric in $\mathcal{H}$ with unit ball D. Since $P_D \leqslant C_K P_K$ from (2.20), the perimeters $P_D(E_i,\Omega')$ are uniformly bounded on every relatively compact set $\Omega' \subseteq \Omega$. Hence the result follows from Remark 2.8 and Theorem 3.7 in Miranda [24] and the fact that $P_K \leqslant C_K P_D$. □

In the statement of the next two theorems, recall that D is the unit ball associated to an scalar product in $\mathcal{H}$. Hence $||\cdot||_D$ induces a distance with associated open balls $B_D(x,r)$ centered at $x \in \mathbb{G}$ of radius $r > 0$.

Theorem 2.11 (Localization Lemma). *Let $E \subset \mathbb{G}$ be a set of finite K-perimeter in $\mathbb{G}$ and $x \in \mathbb{G}$, and let D be the unit ball for a fixed sub-Riemannian metric on $\mathcal{H}$. Then, for almost every $\rho > 0$, the sets $E \cap B_D(x,\rho)$ and $E \setminus B_D(x,\rho)$ have finite perimeter in $\mathbb{G}$ and we have*

$$P_K(E \setminus B_D(x,\rho), \partial B_D(x,\rho) \leqslant C_K \left.\frac{d}{dr}\right|_{r=\rho} m_E(x,r), \tag{2.21}$$

where $m_E(x,r) = |E \cap B_D(x,r)|$ and C_K is the constant in (2.20).

Proof. By Remark 2.8 and Ambrosio's Localization Lemma, Lemma 3.5 in [1], we have

$$P_D(E \setminus B_D(x,\rho), \partial B_D(x,\rho)) \leqslant \left.\frac{d}{dr}\right|_{r=\rho} m_E(x,r).$$

for almost every $\rho > 0$. Inequality (2.21) then follows from (2.20). □

Theorem 2.12 (Local isoperimetric inequality). *Let $E \subset \mathbb{G}$ be a set of finite K-perimeter, and D the unit disk for a sub-Riemannian metric on $\mathscr{H}$. Then, for any $x \in \mathbb{G}$ and $r > 0$, we have*

$$P_K(E, B_D(x,r)) \geqslant C_P \min\left\{|E \cap B_D(x,r)|, |B_D(x,r) \setminus E|\right\}^{(Q-1)/Q}, \tag{2.22}$$

where the positive constant C_P only depends on D and K.

Proof. We choose a D-orthonormal frame $X_1, \dots, X_d$ of $\mathscr{H}$ and consider the associated Carnot-Carathéodory distance d_D. Let $C_K > 0$ be the constant in (2.20). By Remark 2.9, Theorem 1.18 in [16] and inequality $P_K(E,\cdot) \geqslant C_K^{-1} P_D(E,\cdot)$ we have

$$P_K(E, B_D(x,r)) \geqslant C_K^{-1} P_D(E, B_D(x,r)) \geqslant C_K^{-1} C' \left\{|E \cap B_D(x,r)|, |B_D(x,r) \setminus E|\right\}^{(Q-1)/Q},$$

where $C' > 0$ only depends on K and D. Hence $C_P = C_K^{-1} C'$ only depends on K and D. □

2.5. The isoperimetric profile.

Definition 2.13. Let $\mathbb{G}$ be a Carnot group endowed with a sub-Finsler norm $||\cdot||_K$. The *isoperimetric profile* I_K of $(\mathbb{G}, ||\cdot||_K)$ is the function $I_K : (0,\infty) \to \mathbb{R}^+$ defined by

$$I_K(v) = \inf\left\{P_K(E) : E \subset \mathbb{G}, |E| = v\right\}.$$

Definition 2.14. A measurable set $E \subset \mathbb{G}$ is *K-isoperimetric* if $P_K(E) = I_K(|E|)$.

If a set E is K-isoperimetric then we have

$$P_K(E) \geqslant P_K(F)$$

for any other measurable set $F \subset \mathbb{G}$ such that $|E| = |F|$.

Definition 2.15. A set of finite K-perimeter E in $\mathbb{G}$ is *decomposable* if there exists two disjoint sets $F_1, F_2 \subset \mathbb{G}$ with *positive volume* and finite K-perimeter such that $E = F_1 \cup F_2$ and $P_K(E) = P_K(F_1) + P_K(F_1)$. We say that E is *indecomposable* if it is not decomposable.

One of our main results in these notes is

Theorem 2.16. *Let $\mathbb{G}$ be a Carnot group endowed with a sub-Finsler norm $||\cdot||_K$. Then K-isoperimetric sets exist on $\mathbb{G}$ for any positive volume. Moreover, the isoperimetric profile I_K is strictly concave and any isoperimetric set is essentially bounded and undecomposable.*

The behaviour of the perimeter and volume with respect to the intrinsic dilations of $\mathbb{G}$ immediately implies that the K-isoperimetric profile of a Carnot group satisfies

$$I_K(v) = C\, v^{(Q-1)/Q}, \tag{2.23}$$

where $C \geqslant 0$ is a constant and Q is the homogeneous dimension of $\mathbb{G}$. The fact that the constant C is indeed strictly positive will be obtained in the proof of Theorem 2.16, where we ensure the existence of isoperimetric sets.

3. Auxiliary results

In the proof of the main existence result, Theorem 2.16, we shall need some preliminary results and definitions. In this section $(\mathbb{G}, ||\cdot||_K)$ is a Carnot group with a sub-Finsler structure and D is the closed unit disk associated to a fixed sub-Riemannian metric on $\mathscr{H}$.

3.1. **Concentration of area.** Two important consequences of the local isoperimetric inequality (2.22) are the following

Corollary 3.1. *Let $E \subset \mathbb{G}$ be a set with positive finite K-perimeter and positive volume. Let $m \in (0, |B_D(0,1)|/2)$ such that $|E \cap B_D(p,1)| < m$ for all $p \in \mathbb{G}$. Then there is a constant $C > 0$, depending only on K, D and Q, such that*

$$P_K(E) \geqslant \left(\frac{C}{m}\right)^{1/Q} |E|. \tag{3.1}$$

Proof. We consider a maximal family $\mathscr{A}$ of points in $\mathbb{G}$ such that $d_D(p,p') \geqslant 1/2$ for all $p \neq p'$ in $\mathscr{A}$, and $|E \cap B_D(p,1/2)| > 0$ for all $p \in \mathscr{A}$ (e.g, Lemma B.7.1 in [20]). Then

$$|E \setminus \bigcup_{p\in\mathscr{A}} B_D(p,1)| = 0.$$

Otherwise we could find a point q of density 1 in $E \setminus \bigcup_{p\in\mathscr{A}} B_D(p,1)$ and the family $\mathscr{A}\cup\{q\}$ would deny the maximality of $\mathscr{A}$. So we have

$$\begin{aligned}
|E| &= |E \cap \big(\bigcup_{p\in\mathscr{A}} B_D(p,1)\big)| \\
&\leqslant \sum_{p\in\mathscr{A}} |E \cap B_D(p,1)| \\
&= \sum_{p\in\mathscr{A}} |E \cap B_D(p,1)|^{(Q-1)/Q} |E \cap B_D(p,1)|^{1/Q} \\
&\leqslant C_P^{-1}\, m^{1/Q} \sum_{p\in\mathscr{A}} P_K(E \cap B_D(p,1)),
\end{aligned}$$

where the last inequality follows since $|E \cap B_D(p,1)| < m \leqslant |B_D(0,1)|/2$ for all $p \in \mathbb{G}$ and the local isoperimetric inequality (2.22) in $\mathbb{G}$. The constant $C_P > 0$ is the one appearing in (2.22), which only depends on K and D.

To complete the proof we only need to control the overlap of the balls $B_D(p,1)$ when $p \in \mathscr{A}$. Let $z \in \bigcup_{p\in\mathscr{A}} B_D(p,1)$ and let

$$\mathscr{A}(z) = \big\{p \in \mathscr{A} : d_D(z,p) < 1\big\}.$$

The balls $B_D(p,1/4)$, $p \in \mathscr{A}$, are disjoint because of condition $d_D(p,p') \geqslant 1/2$ for all distinct $p, p' \in \mathscr{A}$. Since $B_D(p,1/2) \subset B_D(z, 1+1/2)$ for all $p \in \mathscr{A}(z)$ we get

$$\#\mathscr{A}(z)\, \omega_D \left(\frac{1}{2}\right)^Q \leqslant \omega_D \left(\frac{3}{2}\right)^Q,$$

and so $\#\mathscr{A}(z) \leqslant 3^Q$. Hence

$$|E| \leqslant 3^Q\, C_P^{-1}\, m^{1/Q}\, P_K(E),$$

as claimed. □

Corollary 3.2 (Isoperimetric inequality for small volumes)**.** *There exists $v_0 > 0$ and a positive $C_I > 0$, only depending on K, D and Q, such that*

$$P_K(E) \geqslant C_I\, |E|^{(Q-1)/Q}$$

for any measurable set of volume $0 < |E| < v_0$.

Proof. Let $v_0 = |B_D(0,1)|/2$ and let $E \subset \mathbb{G}$ a measurable set of volume $|E| < v_0$. Consider again a maximal family $\mathscr{A}$ of points in $\mathbb{G}$ such that $d_D(p,p') \geqslant 1/2$ for all distinct $p, p' \in \mathscr{A}$ and $|E \cap B_D(p,1)| > 0$ for all $p \in \mathscr{A}$.

As in the proof of Corollary 3.1, we can bound the overlapping of the balls $B_D(p,1)$ from above by 3^Q, so that, by the local isoperimetric inequality (2.22) we get

$$3^Q P_K(E) \geqslant \sum_{p\in\mathscr{A}} P_K(E, B_D(p,1)) \geqslant C_P \sum_{p\in\mathscr{A}} |E \cap B_D(p,1)|^{(Q-1)/Q} \geqslant C_P\,|E|^{(Q-1)/Q}.$$

The last inequality follows, as in the proof of Corollary 3.1, since the measure of the set $E \setminus \bigcup_{p\in\mathscr{A}} B_D(p,1)$ is equal to 0. To complete the proof we simply take $C_I = C_P/3^Q$. □

3.2. **Boundedness of isoperimetric sets.** We start this section with a classical result. See §4.4.2 in [31] and the references therein.

Theorem 3.3. *Let $E \subset \mathbb{G}$ be a K-isoperimetric set. Then E is essentially bounded.*

Proof. Let $v = |E|$. Assume that E is not bounded. This means that the decreasing function $m(r) = |E \setminus B_D(0,r)|$ is *positive* for all $r > 0$. For every $r > 0$ we consider the set

$$E(r) = E \cap B_D(0,r),$$

which has finite K-perimeter in $\mathbb{G}$, and the intrinsic dilation of ratio $\lambda(r)$ so that $|\lambda(r)E(r)| = |E|$. This implies

$$\lambda(r)^Q = \frac{|E|}{|E(r)|} = \frac{v}{v - m(r)}.$$

Since E is isoperimetric we have

$$P_K(E) \leqslant P_K(\lambda(r)E(r)). \tag{3.2}$$

As E has finite volume, we have $m(r) \to 0$ when $r \to \infty$. By standard properties of sets of finite perimeter, (2.17) and (2.21), we get

$$\lambda(r)^{-(Q-1)} P_K(\lambda(r)E(r)) \leqslant P_K(E) - P_K(E \setminus B_D(0,r)) - 2C_K m'(r)$$

for almost everywhere $r > 0$. From this inequality, the isoperimetric inequality for small volumes applied to $E(r)$ and (3.2) we get

$$-2C_K m'(r) \geqslant C_I m(r)^{(Q-1)/Q} - P_K(E)\Big(1 - \Big(\frac{v - m(r)}{v}\Big)^{(Q-1)/Q}\Big).$$

When $m(r)$ is small enough, there is a positive constant $C > 0$ such that the last summand is bigger than or equal to $-Cm(r)$. Hence we get

$$\begin{aligned} -2C_K m'(r) &\geqslant C_I m(r)^{(Q-1)/Q} - Cm(r) = C_I m(r)^{(Q-1)/Q}\big(1 - \frac{C}{C_I} m(r)^{1/Q}\big) \\ &\geqslant \frac{C_I}{2} m(r)^{(Q-1)/Q}, \end{aligned}$$

for $r > 0$ large enough. Hence, as $m(r) > 0$ for all $r > 0$ we have, for almost everywhere $r > 0$,

$$-\big(m(r)^{1/Q}\big)' = Q^{-1} m(r)^{(1-Q)/Q} m'(r) \geqslant Q^{-1}\frac{C_I}{2C_K} = C > 0$$

and, as $m(r)^{1/Q}$ is a decreasing function, we have, for $a < b$ large enough

$$-(m(b)^{1/Q} - m(a)^{1/Q}) \geqslant -\int_a^b (m(r)^{1/Q})' dr \geqslant C(b-a).$$

This forces $m(b)$ to be negative for b large enough, which provides a contradiction. □

3.3. **Behaviour of minimizing sequences.** In this section we define minimizing sequences for given volume and prove some of their main properties. When working with an isoperimetric set E, we often make some geometric construction removing or adding some small volume from E to obtain a new set F. In order to compare the perimeters of E and F using the isoperimetric property it is necessary to apply a new transformation to F to obtain a new set G satisfying $|E| = |G|$. In these notes the second deformation will be obtained from the intrinsic dilations of the group. In some other cases, deformations by vector fields [31], or by employing Cheeger sets [30], have been used.

Definition 3.4. Given $v > 0$, a sequence $\{E_i\}_{i\in\mathbb{N}}$ of measurable sets with finite K-perimeter is a *minimizing sequence* of volume v if

1. $|E_i| = v$ for all $i \in \mathbb{N}$, and
2. $\lim_{i\to\infty} P_K(E_i) = I_K(v)$.

In non-compact spaces, isoperimetric sets for some given volume do not necessarily exist. For instance, in a Riemannian plane of revolution with increasing Gauss curvature, there are no isoperimetric sets for any value of the area, see Theorem 2.27 in [31]. The reader can also consult [5], a recent result on non existence of isoperimetric sets. However, in general, in Riemannian homogeneous spaces, isoperimetric sets do exist indeed, see §4.4 in [31] or [15]. Thus the convergence behavior of minimizing sequences must be analyzed carefully.

A first refinement we shall use later is that we can always find a minimizing sequence composed of bounded sets.

Lemma 3.5. *Let $(\mathbb{G}, ||\cdot||_K)$ be a sub-Finsler Carnot group, and let $v > 0$. Then there exists a minimizing sequence of volume v composed of bounded sets. In particular, for any $\delta > 0$ we can find a set of volume v and finite K-perimeter such that $P_K(F) \leqslant I_K(v) + \delta$.*

Proof. Let D the closed unit disk associated to a fixed sub-Riemannian metric in $\mathscr{H}$. We take a minimizing $\{E_i\}_{i\in\mathbb{N}}$ sequence of volume v. Since each set E_i has finite volume, we choose a family of increasing radii $\{r_i\}_{i\in\mathbb{N}}$ such that $r_{i+1} - r_i \geqslant i$ and $|E_i \setminus B_D(r_i)| \leqslant 1/i$ for all $i \in \mathbb{N}$. Here $B_D(t) = B_D(0,t)$. Let $m_i(t) = |E_i \cap B_D(t)|$. Since m_i is increasing we have

$$\int_{r_i}^{r_{i+1}} m_i'(t)dt \leqslant v.$$

and so the set of points $t \in [r_i, r_{i+1}]$ where $m_i'(t)$ exists and satisfies $m_i'(t) \leqslant 2v/i$ has positive measure. Hence, for each $i \in \mathbb{N}$ we can find $s_i \in (r_{i+1}, r_i)$ such that $m_i'(s_i)$ exists and satisfies

$$m_i'(s_i) \leqslant \frac{2v}{i}.$$

Now we take $F_i = \lambda_i(E_i \cap B_D(s_i))$, where λ_i is computed so that $|F_i| = v$. As

$$v - |E_i \cap B_D(s_i)| = |E_i| - |E_i \cap B_D(s_i)| = |E_i \setminus B_k(s_i)| \leqslant |E_i \setminus B_D(r_i)| \leqslant \frac{1}{i}$$

and $|E_i \cap B_D(s_i)| \leqslant v$, we get $\lim_{i\to\infty} |E_i \cap B_D(s_i)| = v$ and so $\lim_{i\to\infty} \lambda_i = 1$.

Then we have, by (2.17) and Theorem 2.11,

$$P_K(F_i) = \lambda_i^{Q-1} P_K(E_i \cap B_D(s_i)) \leqslant \lambda_i^{Q-1}\big(P_K(E_i, B_D(s_i)) + C_K m_i'(s_i)\big) \\ \leqslant \lambda_i^{Q-1}\big(P_K(E_i) + C_K m_i'(s_i)\big).$$

Taking lim sup we obtain

$$\limsup_{i\to\infty} P_K(F_i) \leqslant \limsup_{i\to\infty} P_K(E_i) = I_K(v).$$

On the other hand $P_K(F_i) \geqslant I_K(v)$ and so $\lim_{i\to\infty} P_K(F_i) = I_K(v)$. We conclude that $\{F_i\}_{i\in\mathbb{N}}$ is a minimizing sequence of volume v and it is composed of bounded sets. The last assertion in the statement of the Lemma follows immediately. □

Theorem 3.6. *Let $(\mathbb{G}, ||\cdot||_K)$ be a sub-Finsler Carnot group, and let $v > 0$. Consider a minimizing sequence $\{E_i\}_{i\in\mathbb{N}}$ of volume $v > 0$. Then we can find sequences $\{E_i^c\}_{i\in\mathbb{N}}, \{E_i^d\}_{i\in\mathbb{N}}$ such that*

1. *A non-relabeled subsequence of $\{E_i\}_{i\in\mathbb{N}}$ converges in $L^1_{loc}(\mathbb{G})$ to a set $E \subset \mathbb{G}$ with finite volume $|E| \leqslant v$ and finite K-perimeter.*
2. *The sequence $\{E_i^c\}_{i\in\mathbb{N}}$ converges in $L^1_{loc}(\mathbb{G})$ to E and $\lim_{i\to\infty} |E_i^c| = |E|$.*
3. $\lim_{i\to\infty} \big(|E_i^c| + |E_i^d|\big) = v$.
4. *If $|E| > 0$ then the set E is isoperimetric for its volume.*
5. $\liminf_{i\to\infty} P_K(E_i^c) = P_K(E) = I_K(|E|)$.
6. $\liminf_{i\to\infty} P_K(E_i^d) = I_K(v - |E|)$.
7. $I_K(v) = I_K(|E|) + I_K(|E| - v)$.

Proof. We take a sub-Riemannian metric in $\mathcal{H}$ and we consider the associated closed unit disk D. For any $r > 0$, we denote $B_D(r) = B_D(0, r)$.

1. By the Compactness Theorem 2.10, for every $r > 0$ we can extract a subsequence of $\{E_i\}_{i\in\mathbb{N}}$ converging in $L^1(B_D(r))$ to a set of finite perimeter. Choosing a diverging increasing sequence of radii and applying a diagonal argument, we may assume that a non-relabeled subsequence of $\{E_i\}_{i\in\mathbb{N}}$ converges in $L^1_{loc}(\mathbb{G})$ to a set of finite perimeter $E \subset \mathbb{G}$, which might be empty. By Fatou's Lemma,

$$|E| \leqslant \liminf_{i\to\infty} |E_i| \leqslant v.$$

By the lower semicontinuity of the K-perimeter, the set E has finite K- perimeter.

We choose a sequence $\{r_i\}_{i\in\mathbb{N}}$ of increasing radii such that $r_{i+1} - r_i \geqslant i$ for all $i \in \mathbb{N}$ with $r_0 = 0$. Passing again to a subsequence, we may assume

$$\int_{B_D(r_{i+1})} |\chi_{E_i} - \chi_E|\, d\mathbb{G} < \frac{1}{i},$$

Reasoning as in the proof of Lemma 3.5 we get $s_i \in (r_i, r_{i+1})$ such that $m_i'(s_i) \leqslant 2v/i$ for all $i \in \mathbb{N}$, where $m_i(t) = |E_i \cap B_D(t_i)|$.

We define

$$E_i^c = E_i \cap B_D(s_i),$$
$$E_i^d = E_{i+1} \setminus B_D(s_{i+1}).$$

The set E_i^c is bounded for all $i \in \mathbb{N}$ and the sequence $\{E_i^d\}_{i\in\mathbb{N}}$ is divergent.

2. Given $r > 0$ we have

$$\int_{B_D(s_i)} |\chi_{E_i^c} - \chi_E| \, d\mathbb{G} \leqslant \int_{B_D(r_{i+1})} |\chi_{E_i} - \chi_E| \, d\mathbb{G} < \frac{1}{i}$$

This implies $\lim_{i\to\infty} |E_i^c| = |E|$.

3. It follows from the equality $|E_{i+1}^c| + |E_i^d| = v$ and 2.

4. Assume that the set E is not isoperimetric for its volume. Then $I_K(|E|) < P_K(E)$. By Lemma 3.5 there exists a bounded set $F \subset \mathbb{G}$ of volume $|E|$ such that $I_K(|E|) < P_K(F) < P_K(E) - \rho$ for some $\rho > 0$. For i large enough the sets F and E_i^d are disjoint and $|F \cup E_i^d| = |F| + |E_i^d| \to |E| + (v - |E|) = v$ by 2 and 3. Then there is a sequence λ_i converging to 1 such that $|\lambda_i(F \cup E_i^d)| = v$ and

$$\begin{aligned} P_K(\lambda_i(F \cup E_i^d)) &= \lambda_i^{Q-1}\big(P_K(F) + P_K(E_i^d)\big) < \lambda_i^{Q-1}\big(P_K(E) - \rho + P_K(E_i^d)\big) \\ &\leqslant \lambda_i^{Q-1}\big(\liminf_{i\to\infty} P_K(E_i^c) - \rho + P_K(E_i^d)\big). \end{aligned}$$

Hence the K-perimeters of a subsequence of $\{\lambda_i(F \cup E_i^d)\}_{i\in\mathbb{N}}$ converge to $I_K(v) - \rho$, thus providing a contradiction to the fact that $\{E_i\}_{i\in\mathbb{N}}$ is a minimizing sequence for volume v.

5. By the lower semicontinuity of perimeter we have $P_K(E) \leqslant \liminf_{i\to\infty} P_K(E_i^c)$. If the strict equality holds then there is $\rho > 0$ such that $P_K(E) \leqslant \liminf_{i\to\infty} P_K(E_i^c) - \rho$. As E is isoperimetric, it is bounded, so that E and E_i^d are disjoint for large i and $|E \cup E_i^d| \to v$. We take λ_i converging to 1 so that $|\lambda_i(E \cup E_i^d)| = v$. We reason as in 4 so that a subsequence of $\lambda_i(E \cup E_i^d)$ has limit K-perimeter strictly smaller than $I_K(v)$. This contradiction proves $\liminf_{i\to\infty} P_K(E_i^c) = P_K(E) = I_K(v)$.

6. Since $|E_i^d| \to v - |E|$, there is a sequence λ_i converging to 1 such that $|\lambda_i E_i^d| = v - |E|$. Hence $I_K(v - |E|) \leqslant P_K(\lambda_i E_i^d) = \lambda_i^{Q-1} P_K(E_i^d)$ and so

$$I_K(v - |E|) \leqslant \liminf_{i\to\infty} P_K(E_i^d).$$

If we had strict inequality then there would exist a bounded set F of volume $v - |E|$ such that

$$I_K(v - |E|) < P_K(F) \leqslant \liminf_{i\to\infty} P_K(E_i^d) - \rho,$$

for some $\rho > 0$. Since the sets E_i^c and F are bounded we can find left-tranlations ℓ_i such that E_i^c and $\ell_i(F)$ are disjoint. As $|E_i^c \cup \ell_i(F)| = |E_i^c| + |F| \to v$, we can find a sequence λ_i converging to 1 so that $|\lambda_i(E_i^c \cup \ell_i(F))| = v - |E|$. So we have

$$P_K(\lambda_i(E_i^c \cup \ell_i(F))) = \lambda_i^{Q-1}\big(P_K(E_i^c) + P_K(F)\big) \leqslant \lambda_i^{Q-1}\big(P_K(E_i^c) + \liminf_{i\to\infty} P_K(E_i^d) - \rho\big).$$

As in previous cases, the K-perimeters of a subsequence of $\{\lambda_i(E_i^c \cup \ell_i(F))\}_{i\in\mathbb{N}}$ converge to a limit no larger than $I_K(v) - \rho$. This provides a contradiction that shows $I_K(v - |E|) = \liminf_{i\to\infty} P_K(E_i^d)$.

7. It follows from properties 5 and 6. □

4. Proof of Theorem 2.16

Proof of Theorem 2.16. Let $\{E_i\}_{i\in\mathbb{N}}$ be a minimizing sequence for volume $v > 0$. Let us choose $m > 0$ so that

$$m < \min\left\{\frac{1}{v}\frac{|B_D(0,1)|}{2}, \frac{C v^{Q-1}}{(I_K(v) + 1)^Q}\right\},$$

where $C > 0$ is the constant that appears in inequality (3.1). In particular

$$mv < \frac{|B_D(0,1)|}{2}.$$

If $|E_i \cap B_D(p,1)| < mv = m|E_i|$ for all $i \in \mathbb{N}$ and $p \in \mathbb{G}$, then Corollary 3.1 and the choice of m imply

$$P_K(E_i) \geqslant \left(\frac{C}{mv}\right)^{1/Q} |E_i| = \left(\frac{C}{m}\right)^{1/Q} v^{(Q-1)/Q} \geqslant II_K(v) + 1.$$

This leads to a contradiction since $\{E_i\}_{i\in\mathbb{N}}$ is a minimizing sequence. Hence there follows the existence of a non-relabeled subsequence of $\{E_i\}_{i\in\mathbb{N}}$ and points $p_i \in \mathbb{G}$ such that

$$|E_i \cap B_D(p_i, 1)| \geqslant m|E_i|$$

for all $i \in \mathbb{N}$.

Using left-translations, which preserve the volume and the K-perimeter, we can assume that $p_i = 0$ for all $i \in \mathbb{N}$. Hence there is a non-relabeled subsequence converging in $L^1_{loc}(\mathbb{G})$ to a measurable set E with finite perimeter and volume

$$|E| \geqslant |E \cap B_D(0,1)| \geqslant m|E| = mv > 0.$$

In particular, this implies that *the isoperimetric profile function I_K is strictly concave* since $P_K(E) > 0$ and, by the local isoperimetric inequality (2.22) applied at some point of the measure theoretic boundary, the non-negative constant in the expression (2.23) of the isoperimetric profile is strictly positive.

Assume $|E| < v$. Then we have

$$I_K(v) < I_K(|E|) + I_K(v - |E|)$$

since the strict concavity of I_K and the fact that $I_K(0) = 0$ imply

$$\frac{I_K(|E|) - I_K(0)}{|E| - 0} > \frac{I_K(v) - I_K(v - |E|)}{v - (v - |E|)}.$$

But we know from Theorem 3.6 that $I_K(v) = I_K(|E|) + I_K(v - |E|)$. This provides a contradiction that shows $|E| = v$. Hence E is an isoperimetric set of volume v.

The essential boundedness of any isoperimetric set now follows from Theorem 3.3.

To prove the indecomposability of an isoperimetric set $E \subset \mathbb{G}$, we assume that it is decomposable and so we can find two disjoint sets F_1, F_2 of positive volume and finite K-perimeter such that $E = F_1 \cup F_2$ and $P_K = P_K(F_1) + P_K(F_2)$. Letting $v_i = |F_i|$ for $i = 1, 2$, we have

$$I_K(v_1 + v_2) = P_K(E) = P_K(F_1) + P_K(F_2) \geqslant I_K(v_1) + I_K(v_2).$$

But this inequality cannot hold since the strict concavity of I_K together with $I_K(0) = 0$ imply as above

$$\frac{I_K(v_1) - I_K(0)}{v_1 - 0} > \frac{I_K(v_1 + v_2) - I_K(v_2)}{(v_1 + v_2) - v_1}$$

and so

$$I_K(v_1 + v_2) < I_K(v_1) + I_K(v_2).$$

This contradiction implies that E is indecomposable. □

5. Density estimates for isoperimetric sets

In this section we prove that an isoperimetric set coincides essentially with the set of its density 1 points. The discussion follows the one in §5 in Leonardi and Rigot [21].

We consider the function h defined on $\mathbb{G}\times(0,\infty)$ by

$$h(x,r)=r^{-Q}\min\big\{|B_D(x,r)\setminus E|,|B_D(x,r)\cap E|\big\}.$$

Lemma 5.1. *Let $(\mathbb{G},||\cdot||_K)$ be a Carnot group endowed with a sub-Finsler structure, and D the closed unit disk associated to a sub-Riemannian metric on $\mathcal{H}$. Let $E\subset\mathbb{G}$ be a K-isoperimetric set of volume $v>0$. Take any $\varepsilon>0$ satisfying*

$$\varepsilon<\min\Big\{v,\Big(\frac{C_I}{4QC_K}\Big)^Q,\Big(v\frac{C_I}{2}\Big)^Q\Big\}. \tag{5.1}$$

Then, if $h(x,r)\leqslant\varepsilon$ and $0<r\leqslant 1$, then

$$|B_D(x,r/2)\setminus E|=0\quad\text{or}\quad|B_D(x,r)\cap E|=0.$$

Proof. We consider two cases.

Assume first that $h(x,r)=r^{-Q}|B_D(x,r)\setminus E|$. By hypothesis $|B_D(x,r)\setminus E|\leqslant\varepsilon r^Q$ is very small. We define the set $E_t=E\cup B_D(x,t)$ and the increasing volume function

$$m(t)=|B_D(x,t)\setminus E|.$$

For every $t>0$ consider $\lambda(t)>0$ so that $\lambda(t)E_t$ has volume v. This implies

$$\lambda(t)=\Big(\frac{v}{v+m(t)}\Big)^{1/Q}<1.$$

Since E and $\lambda(t)E_t$ have volume v and E is isoperimetric, from standard properties of the K-perimeter there follows, for almost everywhere $t>0$,

$$\begin{aligned}P_K(E)\leqslant P_K\big(\lambda(t)E_t\big)&=\lambda(t)^{Q-1}P_K(E_t)\\&\leqslant\lambda(t)^{Q-1}\big(P_K(E)-P_K(B_D(x,t)\setminus E)+2C_Km'(t)\big).\end{aligned}$$

As $|B_D(x,t)\setminus E|\leqslant\varepsilon r^Q\leqslant\varepsilon<v_0$ and $\lambda(t)<1$ we obtain from the isoperimetric inequality for volumes no larger than v_0,

$$2C_Km'(t)\geqslant C_Im(t)^{(Q-1)/Q}+\big(\lambda(t)^{-(Q-1)}-1\big)P_K(E)\geqslant C_Im(t)^{(Q-1)/Q}.$$

If $m(r/2)>0$ then $m(t)>0$ for all $t\in[r/2,r]$. In this interval the function $m(t)^{1/Q}$ is increasing and, since it does not vanish ay any point, we have for almost everywhere t

$$(m^{1/Q})'(t)=\frac{1}{Q}\,m'(t)\,m(t)^{(1-Q)/Q}\geqslant\frac{C_I}{2QC_K}.$$

So we have

$$\frac{C_Ir}{2QC_K}\leqslant\int_{r/2}^r(m^{1/Q})'(t)dt\leqslant m^{1/Q}(r)-m^{1/Q}(r/2)<m^{1/Q}(r)\leqslant\varepsilon^{1/Q}r,$$

which contradicts the choice of ε. Hence $m(r/2)=0$.

Assume now that $h(x,r)=r^{-Q}|B_D(x,r)\cap E|$. In this case the volume of E inside the ball $B_D(x,t)$ is very small compared to the total volume of $B_D(x,t)$. Let

$$m(t)=|E\cap B_D(x,t)|,$$

and take the set $E^t = \lambda(t)(E \setminus B_D(x,t))$, where

$$\lambda(t) = \left(\frac{v}{v - m(t)}\right)^{1/Q} > 1$$

is computed so that $|E^t| = |E| = v$. So we have, for almost everywhere $t > 0$,

$$\begin{aligned} P_K(E) \leqslant P_K(E^t) &= \lambda(t)^{Q-1} P_K(E^t) \\ &\leqslant \lambda(t)^{Q-1}\big(P_K(E) - P_K(E \cap B_D(x,t)) + 2C_K m'(t)\big). \end{aligned}$$

As $|E \cap B_D(x,t)| \leqslant \varepsilon r^Q < v_0$ we get, from the isoperimetric inequality for small volumes,

$$2C_K m'(t) \geqslant C_I m(t)^{(Q-1)/Q} + \big(\lambda(t)^{1-Q} - 1\big) P_K(E).$$

Note that

$$\lambda(t)^{1-Q} - 1 = \left(\frac{v - m(t)}{v}\right)^{(Q-1)/Q} - 1 < 0.$$

The function $f : [0,v] \to \mathbb{R}$ defined by

$$f(x) = \left(\frac{v - x}{v}\right)^{(Q-1)/Q} - 1$$

satisfies $f''(x) < 0$ in $(0,v)$, and so $f(x) \geqslant -(1/v)x$ in the interval $[0,v]$. So we get

$$2C_K m'(t) \geqslant m(t)^{(Q-1)/Q}\Big(C_I - \frac{1}{v} m(t)^{1/Q}\Big) \geqslant \frac{C_I}{2} m(t)^{(Q-1)/Q}$$

whenever

$$m(t)^{1/Q} \leqslant v\frac{C_I}{2},$$

and this last inequality holds because of out choice of ε Finally we reason as in the previous case: if $m(r/2) > 0$ then we get $(m^{1/Q})'(t) \geqslant C_I/(4QC_K)$ for almost everywhere t, and so

$$\frac{C_I r}{4QC_K} \leqslant \int_{r/2}^{r} (m^{1/Q})'(t)dt \leqslant m^{1/Q}(r) \leqslant \varepsilon^{1/Q} r,$$

providing a contradiction because of our choice of ε. □

Given an isoperimetric set $E \subset \mathbb{G}$ of volume $v > 0$ we now define the sets

$$\begin{aligned} E_1 &= \big\{x \in \mathbb{G} : \exists\, r > 0 \text{ with } |B_D(x,r) \setminus E| = 0\big\}, \\ E_0 &= \big\{x \in \mathbb{G} : \exists\, r > 0 \text{ with } |B_D(x,r) \cap E| = 0\big\}, \\ S &= \big\{x \in \mathbb{G} : h(x,r) > \varepsilon \text{ for all } r \leqslant 1\big\}, \end{aligned}$$

where $\varepsilon > 0$ is one defined in (5.1). With these definitions we have

Theorem 5.2. *Let $E \subset \mathbb{G}$ be a K-isoperimetric set of volume $v > 0$.*

1. *E_1, E_0, S form a partition of $\mathbb{G}$,*
2. *E_0 and E_1 are open sets,*
3. *E_1 coincide with the set of Lebesgue points of E, and E_0 with the set of points of E of density 0,*
4. *$S = \partial E_0 = \partial E_1$*

5. *For any $x \in S$ here is a constant $C > 0$ so that*

$$C^{-1}r^{Q-1} \leqslant P_K(E, B_D(x,r)) \leqslant Cr^{Q-1}$$

for all $x \in S$ and $0 < r < 1$. The constant C depends on the constant $\varepsilon > 0$ in (5.1), the Poincaré constant C_P in (2.22), the homogeneous dimension Q and the constant C_K defined in (2.20).

Proof. The proof of 1-4 follows closely that of Theorem 5.3 in [21] and is straightforward. As for 5, we have, for any $x \in S$ and $r \in (0,1)$, that $h(x,r) > \varepsilon$. By the local isoperimetric inequality (2.22) we obtain the lower bound

$$P_K(E, B_D(x,r)) \geqslant C_P\big(r^Q h(x,r)\big)^{(Q-1)/Q} \geqslant C_P\, \varepsilon^{(Q-1)/Q} r^{Q-1}.$$

To obtain the upper inequality we use

$$P_K(E \cup B_D(x,r)) + P_K(E \cap B_D(x,r)) \leqslant P_K(E) + P_K(B_D(x,r)).$$

As the isoperimetric profile I_K is increasing we have

$$P_K(E \cup B_D(x,r)) \geqslant I_K(|E \cup B_D(x,r)|) \geqslant I_K(|E|) = P_K(E).$$

Hence

$$P_K(E \cap B_D(x,r)) \leqslant P_K(B_D(x,r)).$$

Now the proof is complete since, by (2.16), we obtain

$$P_K(B_D(x,r)) \leqslant C_K P_D(B_D(x,r) \leqslant QC_K|B_D(0,1)|r^{Q-1}$$

and $P_K(E, B_D(x,r)) \leqslant P_K(E \cap B_D(x,r))$. □

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