Contemporary Mathematics

Variational problems related to the sub-Finsler area in the first Heisenberg group $\mathbb{H}^1$

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Contents

1. Introduction

In these notes we consider critical points of the perimeter associated to an asymmetric sub-Finsler structure in the first Heisenberg group $\mathbb{H}^1$. Such a structure is defined by means of an asymmetric left-invariant norm $||\cdot||$ on the horizontal distribution $\mathcal{H}$ of $\mathbb{H}^1$. If we fix any frame of left-invariant horizontal vector fields, any left-invariant norm is uniquely determined by a convex body (compact convex set with non-empty interior) $K \subset \mathbb{R}^2$ containing 0 in its interior. We write $||\cdot||_K$ to indicate the dependence of the norm on K. The case of a symmetric norm corresponds to a centrally symmetric convex body (i.e, such that $K = -K$). The norm associated to the closed unit disc D centered at 0 is the standard Euclidean norm and is denoted by $|\cdot|$. Symmetric sub-Finsler structures in $\mathbb{H}^1$ have received intense interest recently, specially the study of geodesics [**3, 2**], see [**60**] for the classical sub-Riemannian case, and the associated Minkowski content [**79, 80**]. General asymmetric sub-Finsler structures have an associated asymmetric distance and might have different metric properties, see [**58, 59**] and [**18**].

On $\mathbb{H}^1$ we always consider the standard basis of left-invariant vector fields

$$X = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

2000 *Mathematics Subject Classification.* 53C17, 49Q20.

Key words and phrases. Sub-Finsler geometry, Heisenberg group.

The author has been supported by research grant PID2023-151060NB-I00, funded by MCIN/AEI/ 10.13039/501100011033 and by ERDF/EU.

and the left-invariant Riemannian metric g, also denoted by $\langle\cdot,\cdot\rangle$, making X, Y, T orthonormal. The associated Riemannian measure is the Haar measure of the group, and coincides with the Lebesgue measure of the underlying Euclidean space $\mathbb{R}^3$. The measure of a set E is the volume of the set and is denoted by $|E|$. The volume element is denoted by $d\mathbb{H}^1$. Any C^1 surface interacts with horizontal distribution $\mathcal{H}$. The *singular part* of S is the set $S_0 \subset S$ of points $p \in S$ such that $T_pS = \mathcal{H}_p$.

On $\mathbb{H}^1$ there is a one-parameter family of dilations defined by

$$h_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t),$$

for any $(x, y, t) \in \mathbb{H}^1$ and $\lambda > 0$.

Given a left-invariant norm $||\cdot||_K$, a measurable set $E \subset \mathbb{H}^1$ and an open set $\Omega \subset \mathbb{H}^1$, we define the *sub-Finsler perimeter* of E in Ω by

$$P_K(E, \Omega) = \sup\left\{\int_E \operatorname{div} U \, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(\Omega), ||U||_{K,\infty} \leqslant 1\right\}, \tag{1.1}$$

where $\mathcal{H}_0^1(\Omega)$ is the set of C^1 horizontal vector fields with compact support in Ω and $||\cdot||_{K,\infty}$ is the infinity norm associated to $||\cdot||_K$. The perimeter associated to the Euclidean norm $|\cdot|$ is the sub-Riemannian perimeter as it defined in [**44, 36, 35**]. A set has finite perimeter for a given norm if and only if it has finite perimeter for the standard sub-Riemannian perimeter. Hence all known structure results in the standard case apply to the sub-Finsler perimeter, see Franchi et al. [**36**].

In case the boundary S of E is a C^1 or Euclidean lipschitz surface, the perimeter of E is given by the sub-Finsler area functional

$$A_K(S) = \int_S ||N_h||_{K,*} \, dS, \tag{1.2}$$

where $||\cdot||_{K,*}$ is the dual norm of $||\cdot||_K$, N_h is the orthogonal projection to the horizontal distribution of the Riemannian *outer* unit normal N, and dS is the Riemannian measure on S.

1.1. The first variation. This section is based on [**72**].

If we consider a convex set K with boundary of class C^2_+ (i.e., so that ∂K is of class C^2 and ∂K has positive geodesic curvature everywhere), we may compute the first variation of the area functional associated to a vector field U with *compact support in the regular part* of S to get

$$A_K'(0) = \frac{d}{ds}\Big|_{s=0} A_K(\varphi_s(S)) = \int_S u \left(\operatorname{div}_S \eta_K\right) dS.$$

In this formula $\{\varphi_s\}_{s\in\mathbb{R}}$ is the parameter group of diffeomorphisms associated to U, $u = \langle U, N\rangle$ is the normal component of the variation and $\operatorname{div}_S \eta_K$ is the divergence on S of the vector field $\eta_K = \pi_K(\nu_h)$, where $\nu_h = N_h/|N_h|$ is the horizontal unit normal and π_K is the map projecting any vector $v \neq 0$ to the intersection of the supporting line in the direction of v with $||\cdot||_K = 1$ (the boundary of K). The strict convexity of $||\cdot||_K$ implies that this map is well-defined.

The function $H_K = \operatorname{div}_S \eta_K$ appearing in the first variation of perimeter is called the *mean curvature* of S. Further calculations imply that H_K is equal to $\langle D_Z \eta_K, Z\rangle$, where $Z = -J(\nu_h)$ is the horizontal direction on the regular part of S. Hence the mean curvature function is localized on the horizontal curves of S. It is not difficult to check that a horizontal curve in a surface with mean curvature H_K must satisfy a differential equation depending on H_K. Hence we can reconstruct the regular part of a surface with prescribed mean curvature by taking solutions of this differential equation. Furthermore, we might be able classify surfaces with prescribed mean curvature by classifying solutions of this ordinary differential equation and by looking at the interaction of these curves with the singular set S_0 of S composed of the points where the tangent plane is horizontal, as was done in [**76**] for the standard sub-Riemannian perimeter.

Key observations are that horizontal straight lines are solutions of the differential equation for $H_K = 0$ and that horizontal liftings of the curve $||\cdot||_K = 1$ are solutions for $H_K = 1$. The strict convexity of $||\cdot||_K = 1$ together with the invariance of the equation by left-translations and dilations imply that all solutions are of this type.

Hence, given a convex body $K \subset \mathbb{R}^2$ containing 0 in its interior and its associated left–invariant norm $||\cdot||_K$, we consider the set $\mathbb{B}_K$ obtained as the ball enclosed by the horizontal liftings of all translations of the curve ∂K containing 0. It is not difficult to prove that this way we obtain a topological sphere $\mathbb{S}_K$ with two poles on the same vertical line, that is the union of two graphs, and whose singular set consists of the two poles. Moreover the boundary of $\mathbb{B}_K$ is C^2 outside the poles (indeed C^ℓ if the boundary of K is of class C^ℓ, $\ell \geqslant 2$) and of regularity C^2 around the poles. When $K = D$, these sets were build by P. Pansu [**67**] and are frequently referred to as Pansu spheres. They are of class C^2 but not C^3 near the singular points, see Proposition 3.15 in [**21**] and Example 3.3 in [**76**].

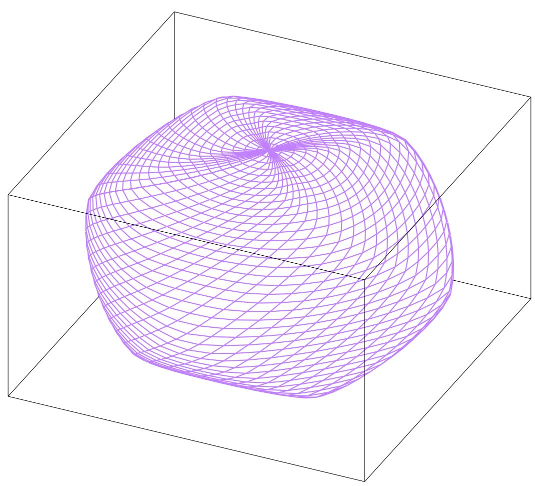


FIGURE 1. The set $\mathbb{B}_K$ when K is the unit ball of the r-norm $||(x,y)||_r = \big(|x|^r + |y|^r\big)^{1/r}$, $r = 1.5$

We observe that these objects have constant mean curvature. Hence they are critical points of the sub-Finsler area functional under a volume constraint. Further evidence that they have stronger minimization properties is given in Section 3.7, where it is proven that, under a geometric condition, a set of finite perimeter E with volume equal to the volume of $\mathbb{B}_K$ has perimeter larger than or equal to the one of the ball $\mathbb{B}_K$. A slightly weaker result for the Euclidean norm was proven in [**74**].

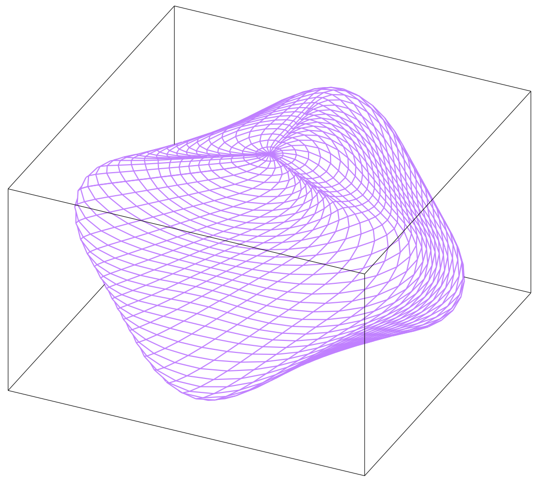


FIGURE 2. The set $\mathbb{B}_K$ when K is a smooth approximation of the triangular norm

We have organized this part into several sections. In section 3.1 we compute the first variation of perimeter for surfaces of class C^2 and, assuming that K is of class C^2_+, prove the property that the regular part of the surface is foliated by horizontal liftings of translations of homothetic expansions of ∂K. In section 3.5 we define the *Pansu-Wulff shapes* and compute some examples and prove regularity properties of these objects. In Section 3.6 we study some geometric properties of the Pansu-Wulff shapes and, finally, in Section 3.7 we obtain a minimization property of these Pansu-Wulff shapes. This property indicates that these shapes are good candidates to be solutions of the sub-Finsler isoperimetric problem in $\mathbb{H}^1$.

Some justification on the terminology *Pansu-Wulff shape* must be given. Consider a norm $||\cdot||$ in Euclidean space and its dual norm $||\cdot||_*$. For a Lipschitz surface S, the integral

$$\int_S ||N||_* dS,$$

where N is an a.e. unit normal to S, defines a functional that represents the Gibbs free energy, proportional to the area of the surface of contact and to the surface tension, of an anisotropic interface separating two fluids or gases. The contribution of each element of area depends on the orientation. An equilibrium state is obtained by minimizing the free energy for a drop of given volume. This is an isoperimetric problem in mathematical terms.

The solutions of this problem were described by the crystallographer G. Wulff in 1895: they are translations and dilations of the set $\{x \in \mathbb{R} : ||x|| \leqslant 1\}$, usually referred to as the *Wulff shape* of the free energy. A first mathematical proof of this fact was given by Dinghas [**25**]. Other versions of Wulff's results were given by Busemann [**7**], Taylor [**81**], Fonseca [**30**] and Fonseca and Müller [**31**]; see also Gardner [**43**], Burago and Zalgaller [**6**], Van Schaftingen [**83**], and Figalli, Maggi and Pratelli [**29**].

The counterpart of the free energy in the Heisenberg group $\mathbb{H}^1$ is given in formula (1.2). When $K = D$ we obtain the classical sub-Riemannian area. In his

Ph. D. Thesis, Pansu exhibited in [**67**] an example of an area-stationary candidate, which coincides with the Pansu-Wulff shape, and conjectured that this set is a solution of the sub-Riemannian isoperimetric problem in $\mathbb{H}^1$. While many partial results have been obtained in the direction of proving this conjecture, see [**75, 76, 74, 63, 32, 62, 61, 55, 21, 33**] and the monograph [**10**], it still remains open.

The Pansu-Wulff shapes were introduced by Pozuelo and Ritoré in [**72**] and also considered by Franceschi et al., see [**34**].

1.2. Regularity of surfaces with prescribed mean curvature. This section is based on [**47**].

The aim of this part is to study the regularity of the characteristic curves of the boundary of a set with continuous prescribed mean curvature in the first Heisenberg group $\mathbb{H}^1$ with a sub-Finsler structure. We assume also in this part that K has C^2 boundary with positive geodesic curvature.

Following De Giorgi [**24**], the authors of [**72**] defined a notion of sub-Finsler K-perimeter, see also [**34**]. Given a measurable set $E \subset \mathbb{H}^1$ and an open subset $\Omega \subset \mathbb{H}^1$, it is said that E has locally finite K-perimeter in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$P_K(E, V) = \sup\left\{ \int_E \operatorname{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), ||U||_{K,\infty} \leqslant 1 \right\} < +\infty,$$

where $\mathcal{H}_0^1(V)$ is the space of horizontal vector fields of class C^1 with compact support in V, and $||U||_{K,\infty} = \sup_{p\in V} ||U_p||_K$. Both the divergence and the integral are computed with respect to a fixed left-invariant Riemannian metric g on $\mathbb{H}^1$. When $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface the K-perimeter coincides with the area functional

$$A_K(S) = \int_S ||N_h||_{K,*}\, d\mathcal{H}^2,$$

where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure associated to the left-invariant Riemannian metric g, N is the outer unit normal to S, defined $\mathcal{H}^2$-a.e on S, N_h is the horizontal projection of N to the horizontal distribution in $\mathbb{H}^1$ and $||\cdot||_{K,*}$ is the dual norm of $||\cdot||_K$.

We say that a set E with Euclidean Lipschitz boundary has *prescribed K-mean curvature* $f \in C^0(\Omega)$ if, for any bounded open subset $V \subset \Omega$, E is a critical point of the functional

$$A_K(S \cap B) - \int_{E\cap B} f\, d\mathbb{H}^1.$$

This notion extends the classical one in Euclidean space and the one introduced in [**42**] for the sub-Riemannian area. We refer the reader to the introduction of [**42**] for a brief historical account and references.

We say that a set E has *constant* prescribed K-mean curvature if there exists $\lambda \in \mathbb{R}$ such that E has prescribed K-mean curvature λ. In Proposition 4.1 we consider a set E with Euclidean Lipschitz boundary and positive K-perimeter. We show that if E is a critical point of the K-perimeter for variations preserving the volume up to first order then E has constant prescribed K-mean curvature on any open set Ω avoiding the singular set S_0 and where $P_K(E, \Omega) > 0$. This result can be applied to isoperimetric regions in $\mathbb{H}^1$ with Euclidean Lipschitz boundary. Critical points of the area have, by definition, prescribed mean curvature equal to 0.

The main result of this paper is Theorem 4.5, where we prove that the boundary S of a set E with prescribed continuous K-mean curvature is foliated by horizontal characteristic curves of class C^2 in its regular part. The minimal assumptions we require for the boundary S of E are to be Euclidean Lipschitz and $\mathbb{H}$-regular. The result holds in particular when the boundary of E is of class C^1. As we point out in Remark 4.9, C^2 regularity is optimal since the Pansu-Wulff shapes obtained in [**72**] have prescribed constant mean curvature and their boundaries are foliated by characteristic curves with the same regularity as that of ∂K, that may be just C^2. In the proof of the Theorem 4.5 we exploit the first variation formula of the area following the arguments developed in [**40, 42**] and make use of the bi-Lipschitz homeomorphism considered in [**65**]. One of the main differences in our setting is that the area functional strongly depends on the inverse π_K of the Gauss map of ∂K. Therefore the first variation of the area depends on the derivative of the map that describes the boundary ∂K. In order to use the bootstrap regularity argument in [**40, 42**] we need to invert this map on the boundary ∂K, that is possible since the geodesic curvature of ∂K is strictly positive, see Lemma 4.6. Moreover, the C^2 regularity of the characteristic curves implies that, on characteristic curves of a boundary with prescribed continuous K-mean curvature f, the ordinary differential equation

$$\langle D_Z\, \pi_K(\nu_h), Z\rangle = f, \tag{1.3}$$

is satisfied. In this equation $\nu_h = N_h/|N_h|$ is the classical sub-Riemannian horizontal unit normal, Z is the unit characteristic vector field tangent to the characteristic curves and D the Levi-Civita connection associated to the left-invariant Riemannian metric g on $\mathbb{H}^1$. Equation (1.3) was proved to hold for C^2 surfaces in [**72**]. For regularity assumptions below $\mathbb{H}$-regular and Euclidean Lipschitz, equation (1.3) holds in a suitable weak sense, a result proved in [**1**] for the sub-Riemannian area, when K coincides with the unit disk centered at 0.

Moreover, in Proposition 4.13 we stress that equation (1.3) is equivalent to

$$H_D = \kappa(\pi_K(\nu_h))f, \tag{1.4}$$

where $H_D = \langle D_Z\nu_h, Z\rangle$ is the classical sub-Riemannian mean curvature introduced in [**1**] and κ is the strictly positive Euclidean curvature of the boundary ∂K. A key ingredient to obtain equation (1.4) is Lemma 4.14, that exploits the ideas of Lemma 4.6 in an intrinsic setting.

This part is a natural continuation of the many recent papers concerning sub-Riemannian area minimizers [**44, 19, 14, 12, 10, 22, 4, 1, 51, 52, 53, 76, 54, 32, 8, 17, 55, 62, 16, 45, 11**]. The sub-Riemannian perimeter functional is a particular case of the sub-Finsler functionals considered in these notes where the convex set is the unit disk D centered at 0. In the pioneering paper [**44**] N. Garofalo and D. M. Nhieu showed the existence of sets of minimal perimeter in Carnot-Carathéodory spaces satisfying the doubling property and a Poincaré inequality. In [**56**] Leonardi and Rigot showed the existence of isoperimetric sets in Carnot groups. However the optimal regularity of the critical points of these variational problems involving the sub-Riemannian area is not completely understood. Indeed, even in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ there are several examples of non-smooth area minimizers: S. D. Pauls in [**69**] exhibited a solution of low regularity for the Plateau problem with smooth boundary datum; on the other hand in [**14, 73, 49**] the authors provided solutions of Bernstein's problem in $\mathbb{H}^1$ that are only Euclidean Lipschitz.

In [66] P. Pansu conjectured that the boundaries of isoperimetric sets in $\mathbb{H}^1$ are given by the surfaces now called Pansu's spheres, union of all sub-Riemannian geodesics of a fixed curvature joining two point in the same vertical line. This conjecture has been solved only assuming *a priori* some regularity of the minimizers of the area with constant prescribed mean curvature. In [76] the authors solved the conjecture assuming that the minimizers of the area are of class C^2, using the description of the singular set, the characterization of area-stationary surfaces, and the ruling property of constant mean curvature surfaces developed in [12]. Hence the *a priori* regularity hypothesis are central to study the sub-Riemannian isoperimetric problem. Motivated by this issue, it was shown in [15] that a C^1 boundary of a set with continuous prescribed mean curvature is foliated by C^2 characteristic curves. Regularity results for Lipschitz viscosity solutions of the minimal surface equation were obtained in [8]. Furthermore, in [42] the authors generalized the previous result when the boundary S is immersed in a three-dimensional contact sub-Riemannian manifold. Finally M. Galli in [40] improved the result in [42] only assuming that the boundary S is Euclidean Lipschitz and $\mathbb{H}$-regular in the sense of [36]. The Bernstein problem in $\mathbb{H}^1$ with Euclidean Lipschitz regularity was treated by S. Nicolussi and F. Serra-Cassano [65]. Partial solutions of the sub-Riemannian isoperimetric problem have been obtained assuming Euclidean convexity [63], or symmetry properties [21, 74, 62, 32]. An analogous sub-Finsler isoperimetric problem might be considered. Candidate solutions would be the Pansu-Wulff shapes considered in [72]. See [72, 34] for partial results in the sub-Finsler isoperimetric problem and [79] for earlier work.

We have organized this part into two sections. Section 4.2 is dedicated to the proof of the main Theorem 4.5, that ensures that the characteristic curves are C^2. Finally in Section 4.3 we deal with the K-mean curvature equation, see Proposition 4.12 and Proposition 4.13.

1.3. Some examples. This section is based on [46].

The regularity of perimeter-minimizing sets in sub-Finsler geometry is currently one of the most challenging problems in Calculus of Variations.

The regularity of sub-Riemannian perimeter-minimizing sets has been investigated by a large number of researchers [12, 76, 20, 4, 23, 53, 38, 77, 39, 73, 15, 14, 65, 8]. The boundaries of the conjectured solutions to the isoperimetric problem are of class C^2, see [10], although there exist examples of area-minimizing horizontal graphs which are merely Euclidean Lipschitz, see [14, 64, 73]. The sub-Riemannian Plateau problem was first considered by Pauls [68]. Afterwards, under given Dirichlet conditions on p-convex domains, Cheng, Hwang and Yang [14] proved existence and uniqueness of t-graphs (horizontal graphs of the form $t = u(x, y)$) which are Lipschitz continuous weak solutions of the minimal surface equation in $\mathbb{H}^1$. Later, Pinamonti, Serra Cassano, Treu and Vittone [70] obtained existence and uniqueness of t-graphs on domains with boundary data satisfying a bounded slope condition, thus showing that Lipschitz regularity is optimal at least in the first Heisenberg group $\mathbb{H}^1$. Capogna, Citti and Manfredini [8] established that the intrinsic graph of a Lipschitz continuous function which is, in addition, a viscosity solution of the sub-Riemannian minimal surface equation in $\mathbb{H}^1$, is of class $C^{1,\alpha}$, with higher regularity in the case of $\mathbb{H}^n$, $n > 1$, see [9]. It was shown in [15] that the regular part of a t-graph of class C^1 with continuous prescribed sub-Riemannian mean curvature in $\mathbb{H}^1$ is foliated by C^2 characteristic curves. Furthermore, in [42] the authors generalized the previous result when the boundary S is a general C^1 surface in a three-dimensional contact sub-Riemannian manifold. Later, Galli in [40] improved the result in [42] only assuming that the boundary S is Euclidean Lipschitz and $\mathbb{H}$-regular. Recently, in [47] the first and third authors extended the result in [40] to the sub-Finsler Heisenberg groups. Up to now, determining the optimal regularity of perimeter-minimizing $\mathbb{H}$-regular hypersurfaces in the Heisenberg group remains an open problem.

Bernstein type problems for surfaces in $\mathbb{H}^1$ have also received a special attention. The nature of the sub-Riemannian Bernstein problem in the Heisenberg group is completely different from the Euclidean one even for graphs. On the one hand the area functional for t-graphs is convex as in the Euclidean setting. Therefore the critical points of the area are automatically minimizers for the area functional. However, since t-graphs admit singular points where the horizontal gradient vanishes their classification is not an easy task. Thanks to a deep study of the singular set for C^2 surfaces in $\mathbb{H}^1$, Cheng, Hwang, Malchiodi, and Yang [12] showed that minimal t-graphs of class C^2 are congruent to a family of surfaces including the hyperbolic paraboloid $u(x, y) = xy$ and the Euclidean planes. Under the hypothesis that the surface is area-stationary, Ritoré and Rosales proved in [76] that the surface must be congruent to a hyperbolic paraboloid or to a Euclidean plane. If we consider the class of Euclidean Lipschitz t-graphs, the previous classification does not hold since there are several examples of area-minimizing surfaces of low regularity, see [73]. The complete classification for C^2 surfaces was established by Hurtado, Ritoré and Rosales in [53], by showing that a complete, orientable, connected, stable area-stationary surface is congruent either to the hyperbolic paraboloid $u(x, y) = xy$ or to a Euclidean plane. As in the Euclidean setting the stability condition is crucial in order to discard some minimal surfaces such as helicoids and catenoids.

On the other hand, the study of the regularity of intrinsic graphs (i. e., Riemannian graphs over vertical planes) is a completely different problem since the area functional for such graphs is not convex. Indeed, Danielli, Garofalo, Nhieu in [20] discovered that the family of graphs

$$u_\alpha(x, t) = \frac{\alpha x t}{1 + 2\alpha x^2}, \quad \alpha > 0,$$

are area-stationary but *unstable*. In [64], Monti, Serra Cassano and Vittone provided an example of an area-minimizing intrinsic graph of regularity $C^{1/2}(\mathbb{R}^2)$ that is an intrinsic cone. Therefore the Euclidean threshold of dimension $n = 8$ fails in the sub-Riemannian setting. In [4], Barone Adesi, Serra Cassano and Vittone classified complete C^2 area-stationary intrinsic graphs. Later Danielli, Garofalo, Nhieu and Pauls in [23] showed that a C^2 complete stable embedded minimal surface in $\mathbb{H}^1$ with empty characteristic set must be a plane. In [41] Galli and Ritoré proved that a complete, oriented and stable area-stationary C^1 surface without singular points is a vertical plane. Later, Nicolussi Golo and Serra Cassano [65] showed that Euclidean Lipschitz stable area-stationary intrinsic graphs are vertical planes. Recently, Giovannardi and Ritoré [48] showed that in the Heisenberg group $\mathbb{H}^1$ with a sub-Finsler structure, a complete, stable, Euclidean Lipschitz surface without singular points is a vertical plane and Young [85] proved that a ruled area-minimizing entire intrinsic graph in $\mathbb{H}^1$ is a vertical plane by introducing a family of deformations of graphical strips based on variations of a vertical curve.

In this note, we provide examples of entire perimeter-minimizing t-graphs for a fixed but arbitrary left-invariant sub-Finsler structure in the first Heisenberg group $\mathbb{H}^1$. Our examples are inspired by the corresponding sub-Riemannian ones in [**73**]. Of particular interest are the conical examples, invariant by the non-isotropic dilations of $\mathbb{H}^1$. In the sub-Riemannian case these examples were investigated in [**49**] and [**73**].

The part is organized the following way. In Theorem 5.1 of Section 5.1 we obtain a necessary and sufficient condition, inspired by Theorem 3.1 in [**72**], for a surface to be a critical point of the sub-Finsler area. We assume that the surface is piecewise C^2, and composed of pieces meeting in a C^1 way along C^1 curves. This condition will allow us to construct area-minimizing examples in Proposition 5.7 of Section 5.2, and examples with low regularity in Proposition 5.8. The same construction, keeping fixed the angle at one side (and hence at the other one) of the singular line, provides examples of area-minimizing cones, see Corollary 5.9. Finally, in Section 5.3 we exhibit some examples of area-minimizing cones in the spirit of [**49**]. These examples are obtained in Theorem 5.14 from circular sectors of the area-minimizing cones with one singular half-line obtained in Corollary 5.9.

2. Preliminaries

2.1. The first Heisenberg group $\mathbb{H}^1$. We denote by $\mathbb{H}^1$ the first Heisenberg group: the 3-dimensional Euclidean space $\mathbb{R}^3$ with coordinates (x, y, t), endowed with the product $*$ defined by

$$(a, b, c) * (x, y, t) = (a + x, b + y, c + t + (-ay + bx).$$

For $p \in \mathbb{H}^1$, the *left translation* by p is the diffeomorphism $L_p(q) = p * q$. A frame of left-invariant vector fields is given by

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

The *horizontal distribution* $\mathcal{H}$ in $\mathbb{H}^1$ is the smooth planar distribution generated by X and Y. The *horizontal projection* of a vector U onto $\mathcal{H}$ will be denoted by U_h. A vector field U is called *horizontal* if $U = U_h$. A *horizontal curve* is a C^1 curve whose tangent vector lies in the horizontal distribution.

We shall consider on $\mathbb{H}^1$ the left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$, so that $\{X, Y, T\}$ is a global orthonormal frame, and let D be the Levi-Civita connection associated to the Riemannian metric g.

We denote by $[U, V]$ the Lie bracket of two C^1 vector fields U, V on $\mathbb{H}^1$. Note that $[X, T] = [Y, T] = 0$, while $[X, Y] = -2T$. The last equality implies that $\mathcal{H}$ is a bracket generating distribution. Moreover, by Frobenius Theorem we have that $\mathcal{H}$ is non-integrable. The vector fields X and Y generate the kernel of the (contact) 1-form $\omega := -y\,dx + x\,dy + dt$.

We shall consider on $\mathbb{H}^1$ the (left-invariant) Riemannian metric $g = \langle \cdot , \cdot \rangle$ so that $\{X, Y, T\}$ is an orthonormal basis at every point, and the associated Levi-Civitá connection D. The modulus of a vector field U with respect to this Riemannian metric will be denoted by $|U|$. The following derivatives can be easily computed

$$\begin{aligned} &D_X X = 0, && D_Y Y = 0, && D_T T = 0, \\ &D_X Y = -T, && D_X T = Y, && D_Y T = -X, \\ &D_Y X = T, && D_T X = Y, && D_T Y = -X. \end{aligned} \tag{2.1}$$

Setting $J(U) = D_U T$ for any vector field U in $\mathbb{H}^1$ we get $J(X) = Y$, $J(Y) = -X$ and $J(T) = 0$. Therefore $-J^2$ coincides with the identity when restricted to the horizontal distribution. The Riemannian volume of a set E is, up to a constant, the Haar measure of the group and is denoted by $|E|$. Since left-translations are affine Euclidean maps with Jacobian 1, there follows that the Haar measure coincides with Lebesque measure in $\mathbb{R}^3$ since left-translations are affine maps with Jacobian 1. More precisely,

$$L_{(a,b,c)} \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & -a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The integral of a function f with respect to the Riemannian measure is denoted by $\int f \, d\mathbb{H}^1$.

We refer to [**76**] for notation and background.

2.2. The pseudo-hermitian connection. The *pseudo-hermitian* connection ∇ on $\mathbb{H}^1$ is the only affine connection satisfying the following properties:

(1) ∇ is a metric connection,
(2) $\operatorname{Tor}(U, V) = 2 \langle J(U), V\rangle T$ for all vector fields U, V.

The existence of the pseudo-hermitian connection can be easily obtained adapting the proof of existence of the Levi-Civita connection, see Koszul formula, Theorem 3.6 in [**26**].

We shall use the following relation between the pseudo-hermitian and the Levi-Civita connections.

LEMMA 2.1. *Let U, V and W be vector fields where V and W are horizontal. Then the following equation holds*

$$\langle \nabla_U V, W\rangle = \langle D_U V, W\rangle + \langle J(W), V\rangle\langle T, U\rangle. \tag{2.2}$$

In particular

$$\nabla_U V = D_U V - \langle T, U\rangle J(V). \tag{2.3}$$

PROOF. We use Koszul formula for ∇, see §3 in [**26**]. The terms in the first two lines are equal to $\langle D_U V, W\rangle$. The last three terms can be computed using the expression for the torsion to get

$$\langle J(W), V\rangle\langle T, U\rangle.$$

This proves (2.2). Equation (2.3) follows since $\langle J(V), W\rangle = -\langle V, J(W)\rangle$. □

Using Koszul formula it can be easily seen that $\nabla X = \nabla Y = 0$.

COROLLARY 2.2. *Let $\gamma : I \to S$ be a curve on $\mathbb{H}^1$ and let ∇/ds, D/ds be the covariant derivatives induced by the pseudo-hermitian connection and the Levi-Civita connection in γ, respectively. Let V be a vector field along γ. Then we* *have*

$$\frac{\nabla}{ds}V = \frac{D}{ds}V - \langle \dot{\gamma}, T\rangle J(V). \tag{2.4}$$

In particular, if γ is a horizontal curve, the covariant derivatives coincide.

2.3. Immersed surfaces in $\mathbb{H}^1$. Following [**1, 36**] we provide the following definition.

DEFINITION 2.3 ($\mathbb{H}$-regular surfaces). A real measurable function f defined on an open set $\Omega \subset \mathbb{H}^1$ is of class $C^1_{\mathbb{H}}(\Omega)$ if the distributional derivative $\nabla_{\mathbb{H}} f = (Xf, Yf)$ is represented by a continuous function. This means that Xf, Yf are continuous functions.

We say that $S \subset \mathbb{H}^1$ is an $\mathbb{H}$-regular surface if for each $p \in \mathbb{H}^1$ there exist a neighborhood U and a function $f \in C^1_{\mathbb{H}}(U)$ such that $\nabla_{\mathbb{H}} f \neq 0$ and $S \cap U = \{f = 0\}$. Then the continuous horizontal unit normal is given by

$$\nu_h = \frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|}.$$

Given an oriented Euclidean Lipschitz surface S immersed in $\mathbb{H}^1$, its unit normal N is defined $\mathcal{H}^2$-a.e. in S, where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure associated to the Riemannian distance induced by g. In case S is the boundary of a set $E \subset \mathbb{H}^1$, we always choose the outer unit normal. We say that a point p belongs to the *singular set* S_0 of S if $p \in S$ is a differentiable point and the tangent space T_pS coincides with the horizontal distribution $\mathcal{H}_p$. Therefore the horizontal projection of the normal N_h at singular points vanishes. In $S \setminus S_0$ the horizontal unit normal ν_h is defined $\mathcal{H}^2$-a.e. by

$$\nu_h = \frac{N_h}{|N_h|},$$

where N_h is the horizontal projection of the normal N. The vector field Z is defined $\mathcal{H}^2$-a.e. on $S \setminus S_0'$ by $Z = -J(\nu_h)$, and it is tangent to S and horizontal.

$\mathbb{H}$-regularity plays an important role in the regularity theory of sets of finite sub-Riemannian perimeter. In [**36**], B. Franchi, R. Serapioni and F. Serra-Cassano proved that the boundary of such a set is composed of $\mathbb{H}$-regular surfaces and a singular set of small measure.

2.4. Sub-Finsler norms. The notion of norm we use in these notes is the one of asymmetric norm. This is a non-negative function $||\cdot|| : V \to \mathbb{R}$ defined on a finite-dimensional real vector space V satisfying

(1) $||v|| = 0$ if and and only if $v = 0$,
(2) $||\lambda v|| = \lambda ||v||$, for all $\lambda \geqslant 0$ and $v \in V$, and
(3) $||v + w|| \leqslant ||v|| + ||w||$, for all $v, w \in V$.

We stress the fact that we are not assuming the symmetry property $||-v|| = ||v||$.

Associated to a given a norm $||\cdot||$ in V we have the set $F = \{u \in V : ||u|| \leqslant 1\}$, which is compact, convex and includes 0 in its interior. Reciprocally, given a compact convex set K with $0 \in \text{int}(K)$, the function $||u||_K = \inf\{\lambda \geqslant 0 : u \in \lambda K\}$ defines a norm in V so that $F = \{u \in V : ||u||_K \leqslant 1\}$. The set F is referred to as the closed unit ball (centered at 0) of the norm $||\cdot||$.

Given a norm $||\cdot||$ and an scalar product $\langle\cdot,\cdot\rangle$ in V, we consider its dual norm $||\cdot||_*$ of $||\cdot||$ with respect to $\langle\cdot,\cdot\rangle$ defined by

$$||u||_* = \sup_{||v||\leqslant 1} \langle u, v\rangle.$$

The dual norm is the support function h of the unit ball $K = \{u \in V : ||u|| \leqslant 1\}$ with respect to the scalar product $\langle\cdot,\cdot\rangle$. From this point on, we assume that $||\cdot||$ is smooth (i.e., it is C^∞ in $V \setminus \{0\}$) and strictly convex:

$$||\lambda u + (1-\lambda)v|| < 1, \quad \text{for all } \lambda \in (0,1), \text{when } u \neq v, ||u|| = ||v|| = 1.$$

Given $u \in V$, the compactness of the unit ball of $||\cdot||$ and the continuity of $||\cdot||$ implies the existence of $u_0 \in V$ satisfying equality $||u||_* = \langle u, u_0\rangle$. Moreover, it can be easily checked that $||u_0|| = 1$. In general, a point u_0 satisfying this property is not unique, but uniqueness follows from the assumption that $||\cdot||$ is strictly convex: this is proved by contradiction assuming the existence of another point u_0' with $||u_0'|| \leqslant 1$ satisfying $||u||_* = \langle u, u_0'\rangle$. Of course u_0' must also satisfy $||u_0'|| = 1$. Then all the points v in the segment $[u_0, u_0']$ satisfy $||v|| \leqslant 1$ and $||u||_* = \langle u, v\rangle$; hence $||v|| = 1$. But this contradicts the strict convexity of $||\cdot||$ unless $u_0 = u_0'$. We shall define $\pi(u)$ as the only vector satisfying $||\pi(u)|| = 1$ and

$$h(u) = ||u||_* = \langle u, \pi(u)\rangle.$$

If $\lambda > 0$ then it is easily checked that $\pi(\lambda u) = \pi(u)$.

We further assume that K is of class C^ℓ_+, with $\ell \geqslant 2$. This means that ∂K is of class C^ℓ, $\ell \geqslant 2$, and that the geodesic curvature of ∂K is everywhere positive. Hence the Gauss map $N : \partial K \to \mathbb{S}^1$ to the unit circle is a diffeomorphism of class $C^{\ell-1}$. Since $\pi = N^{-1}$ we conclude that π is of class $C^{\ell-1}$. Moreover, by Corollary 1.7.3 in [78] we have

$$\nabla h(u) = N^{-1}\bigg(\frac{u}{|u|}\bigg),$$

and so h is of class C^ℓ.

Given a norm $||\cdot||_0$ in $\mathcal{H}_0$, we extend it by left-invariance to a norm $||\cdot||$ in the whole horizontal distribution $\mathcal{H}$ by means of the formula

$$||v||_p = ||d\ell_p^{-1}(v)||_0, \qquad p \in \mathbb{H}^1, v \in \mathcal{H}_p. \tag{2.5}$$

In particular, for a horizontal vector field $fX + gY$, its norm at a point $p \in \mathbb{H}^1$ is given by $||f(p)X_0 + g(p)Y_0||_0$. Identifying the vector $aX_0 + bY_0 \in \mathcal{H}_0$ with the Euclidean vector (a,b), we can define a norm in $\mathbb{R}^2$ by the formula $||(a,b)||_e = ||aX_0 + bY_0||_0$.

We consider the norm $(||\cdot||_0)_*$, dual to $||\cdot||_0$ in $\mathcal{H}_0$, and we extend it by left-invariance to a norm $||\cdot||_*$ in $\mathcal{H}$. It can be easily checked that $(||\cdot||_*)_p$ is the dual norm to $||\cdot||_p$ since

$$\begin{aligned}(||v||_*)_p = (||d\ell_p^{-1}(v)||_0)_* &= \sup_{||w||_0\leqslant 1, w\in\mathcal{H}_0} \langle d\ell_p^{-1}(v), w\rangle \\ &= \sup_{||w'||_p\leqslant 1, w'\in\mathcal{H}_p} \langle v, w'\rangle \\ &= (||v||_p)_*.\end{aligned}$$

When $||\cdot||_0$ is C^l_+ with $l \geqslant 2$, all norms $||\cdot||_p$ are C^l_+. Given a horizontal vector field U of class C^1, we define $\pi(U)$ as the C^1 horizontal vector field satisfying

$$||U||_* = \langle U, \pi(U)\rangle, \tag{2.6}$$

or, equivalently, $(||U_p||_p)_* = \langle U_p, \pi(U)_p\rangle$ for all $p \in \mathbb{H}^1$. We recall that $\pi(fU) = \pi(U)$ for any positive smooth function f.

2.5. The sub-Finsler perimeter. Let $E \subset \mathbb{H}^1$ be a measurable set, $||\cdot||_K$ the left-invariant norm associated to a convex body $K \subset \mathbb{R}^2$ so that $0 \in \text{int}(K)$, and $\Omega \subset \mathbb{H}^1$ an open subset. We say that E has locally finite K-perimeter in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$P_K(E, V) = \sup\left\{ \int_E \text{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}^1_0(V), ||U||_{K,\infty} \leqslant 1 \right\} < +\infty.$$

In this expression, $\mathcal{H}^1_0(V)$ is the space of horizontal vector fields of class C^1 with compact support in V, and $||U||_{K,\infty} = \sup_{p\in V} ||U_p||_K$. The integral is computed with respect to the Riemannian measure $d\mathbb{H}^1$ of this left-invariant metric.

Let K, K' bounded convex bodies containing 0 in its interior. Then there exist constants $\alpha, \beta > 0$ such that

$$\alpha||x||_{K'} \leqslant ||x||_K \leqslant \beta||x||_{K'}, \quad \text{for all } x \in \mathbb{R}^2.$$

Let $E \subset \mathbb{H}^1$ be a measurable set, $\Omega \subset \mathbb{H}^1$ an open set and $V \subset \Omega$ a relatively open set. Take $U \in \mathcal{H}^1_0(V)$ a vector field with $||U||_{K,\infty} \leqslant 1$. Hence $||\alpha U||_{K'} \leqslant ||U||_K \leqslant 1$ and

$$\int_E \text{div}(U) d\mathbb{H}^1 = \frac{1}{\alpha} \int_E \text{div}(\alpha U)\, d\mathbb{H}^1 \leqslant \frac{1}{\alpha} |\partial E|_{K'}(V),$$

Taking supremum over the set of C^1 horizontal vector fields with compact support in V and $||\cdot||_K \leqslant 1$, we get $P_K(E, V) \leqslant \frac{1}{\alpha}|\partial E|_{K'}(V)$. In a similar way we get the inequality $\frac{1}{\beta}|\partial E|_{K'}(V) \leqslant P_K(E, V)$, so that we have

$$\tfrac{1}{\beta}|\partial E|_{K'}(V) \leqslant |\partial E_K|(V) \leqslant \tfrac{1}{\alpha}|\partial E|_{K'}(V). \tag{2.7}$$

As a consequence, E has locally finite K-perimeter if and only if it has locally finite K'-perimeter.

Let $E \subset \mathbb{H}^1$ be a set with locally finite K-perimeter in Ω. Given the standard basis X, Y of the horizontal distribution, we can define a linear functional $L : C^1_0(\Omega, \mathbb{R}^2) \to \mathbb{R}$ by

$$L(g) = L((g_1, g_2)) = \int_E \text{div}(g_1 X + g_2 Y)\, d\mathbb{H}^1.$$

For any relatively compact open set $V \subset \Omega$ we have

$$C(V) := \sup\{L(g) : g \in C^1_0(V, \mathbb{R}^2), ||g||_{K,\infty} \leqslant 1\} < +\infty,$$

We fix any compact subset $C \subset \Omega$ and take a relatively compact open set V such that $C \subset V \subset \Omega$. For each $g \in C_0(\Omega, \mathbb{R}^2)$ with support in K we can find a sequence of C^1 functions $(g_i)_{i\in\mathbb{N}}$ with support in V such that g_i converges uniformly to g. Hence equality

$$\overline{L}(g) = \lim_{i\to\infty} L(g_i)$$

allows to extend L to a linear functional $\overline{L} : C_0(\Omega, \mathbb{R}^2) \to \mathbb{R}$ satisfying

$$\sup\{\overline{L}(g) : g \in C_0(\Omega, \mathbb{R}^2), \text{supp}(g) \subset C, ||g||_{K,\infty} \leqslant 1\} \leqslant C(V) < +\infty.$$

The proof of the Riesz Representation Theorem, see § 1.8 in [27], can be adapted to obtain the existence of a Radon measure μ_K on Ω and a μ_K-measurable horizontal vector field ν_K in Ω so that $\nu_K = \nu_1 X + \nu_2 Y$, with $(\nu_1, \nu_2) : \Omega \to \mathbb{R}^2$ a μ_K-measurable function, satisfying

$$\overline{L}(g) = \int_\Omega \langle g_1 X + g_2 Y, \nu_K \rangle \, d\mu_K.$$

The measure μ_K is the total variation measure

$$\mu_K(V) = \sup\{\bar{L}(g) : g \in C_0(\Omega, \mathbb{R}^2), \operatorname{supp}(g) \subset V, ||g||_{K,\infty} \leqslant 1\}$$

that coincides with $P_K(E, V)$ because $\overline{L}$ is a continuous extension of L. Henceforth we denote μ_K by $|\partial E|_K$.

Let us check that

$$||(\nu_K)_p||_{K,*} = 1 \text{ for } |\partial E|_K\text{-}a.e.\ p. \tag{2.8}$$

Here $||\cdot||_{K,*}$ is the dual norm of $||\cdot||_K$. To prove (2.8) we take a relatively compact open set $V \subset \Omega$ and $g \in C_0(\Omega, \mathbb{R}^2)$ with $\operatorname{supp}(g) \subset V$ and $||g||_{K,\infty} \leqslant 1$. Since $\langle g_1 X + g_2 Y, \nu_K \rangle \leqslant ||\nu_K||_{K,*}$ we have

$$\overline{L}(g) \leqslant \int_V ||\nu_K||_{K,*} d|\partial E|_K.$$

Taking supremum over such g we have

$$|\partial E|_K(V) \leqslant \int_V ||\nu_K||_{K,*} d|\partial E|_K.$$

On the other hand, we can take a sequence of functions $(h_i) = ((h_1)_i, (h_2)_i)$ with support in V such that $||h_i||_K \leqslant 1$ and $\langle (h_1)_i X + (h_2)_i Y, \nu_K \rangle$ converges to $||\nu_K||_{K,*}$ $|\partial E|_K$-a.e. This is a consequence of Lusin's Theorem, see § 1.2 in [27], and follows by approximating the measurable function $\pi_K(\nu_K)$ by continuous uniformly bounded functions. Then we would have

$$\int_V ||\nu_K||_{K,*} d|\partial E|_K = \lim_{i\to\infty} \langle (h_1)_i X + (h_2)_i Y, \nu_K \rangle d|\partial E|_K \leqslant |\partial E|_K(V).$$

So we would have

$$|\partial E|_K(V) = \int_V ||\nu_K||_{K,*} d|\partial E|_K$$

and so $||\nu_K||_{K,*} = 1$ for $|\partial E|_K$-a.e.

Given two convex sets $K, K' \subset \mathbb{R}^2$ containing 0 in their interiors, we shall obtain the following representation formula for the sub-finsler perimeter measure $|\partial E|_K$ and the vector field ν_K

$$|\partial E|_K = ||\nu_{K'}||_{K,*} |\partial E|_{K'}, \quad \nu_K = \frac{\nu_{K'}}{||\nu_{K'}||_{K,*}}. \tag{2.9}$$

From (2.7), there exist two positive constants λ, Λ such that

$$\lambda |\partial E|_K \leqslant |\partial E|_{K'} \leqslant \Lambda |\partial E|_K.$$

This implies that each of the Radon measures $|\partial E|_K, |\partial E|_{K'}$ is absolutely continuous with respect to the other one. Hence both Radon-Nikodym derivatives exist.

Take a relatively compact open set $V \subset \Omega$ and $U \in \mathcal{H}_0^1(V)$. Then we have

$$\begin{aligned}\int_V \langle U, \nu_{K'}\rangle \, d|\partial E|_{K'} &= \int_V \chi_E \operatorname{div}(U)\, d\mathbb{H}^1 \\ &= \int_V \langle U, \nu_K\rangle \, d|\partial E|_K = \int_V \langle U, \frac{d|\partial E|_K}{d|\partial E|_{K'}}\, \nu_K\rangle \, d|\partial E|_{K'}.\end{aligned} \tag{2.10}$$

By the uniqueness of $\nu_{K'}$ we have

$$\nu_{K'} = \frac{d|\partial E|_K}{d|\partial E|_{K'}}\, \nu_K, \quad |\partial E|_{K'}\text{-}a.e. \tag{2.11}$$

On the other hand, inserting $U \in \mathcal{H}_0^1(V)$ in (2.10) with $||U||_K \leqslant 1$ we get

$$\int_V \langle U, \nu_K\rangle d|\partial E|_K = \int_V \langle U, \nu_{K'}\rangle \, d|\partial E|_{K'} \leqslant \int_V ||\nu_{K'}||_{K,*} d|\partial E|_{K'}.$$

Taking supremum over U we obtain

$$\int_V \frac{d|\partial E|_K}{d|\partial E|_{K'}} d|\partial E|_{K'} = |\partial E|_K(V) \leqslant \int_V ||\nu_{K'}||_{K,*} d|\partial E|_{K'}$$

and, since V is arbitrary, we have

$$\frac{d|\partial E|_K}{d|\partial E|_{K'}} \leqslant ||\nu_{K'}||_{K,*} \quad |\partial E|_K\text{-}a.e. \tag{2.12}$$

Substituting (2.11) into (2.12) we have

$$\frac{d|\partial E|_K}{d|\partial E|_{K'}} \leqslant ||\nu_{K'}||_{K,*} = \frac{d|\partial E|_K}{d|\partial E|_{K'}} \quad |\partial E|_K\text{-}a.e.$$

Hence we have equality and so

$$\frac{d|\partial E|_K}{d|\partial E|_{K'}} = ||\nu_{K'}||_{K,*} \quad |\partial E|_K\text{-}a.e. \tag{2.13}$$

Hence we get from equation (2.9) from (2.13) and (2.11).

In the case of a set E with C^1 boundary $S = \partial E$ it is not difficult to check that

$$|\partial E|_K = ||N_h||_{K,*} dS, \quad \nu_K = \frac{N_h}{||N_h||_{K,*}},$$

where N_h is the horizontal projection of the unit normal to S and dS is the Riemannian measure on S. Indeed, for the closed unit disk $D \subset \mathbb{R}^2$ centered at 0 we know that in the C^1 case $\nu_D = \nu_h$ and $|N_h| = ||N_h||_{D,*}$. Hence we have

$$|\partial E|_K = ||\nu_h||_{K,*} d|\partial E|_D, \quad \nu_K = \frac{\nu_h}{||\nu_h||_{K,*}}. \tag{2.14}$$

Here $|\partial E|_D$ is the standard sub-Riemannian measure.

Remark 2.4. Some other notions of perimeter and area for higher codimensional submanifolds have been considered in [**37**, **57**, **45**].

Given two convex sets $K, K' \subset \mathbb{R}^2$ containing 0 in their interiors, we have the following representation formula for the sub-Finsler perimeter measure $|\partial E|_K$ and the vector field ν_K

$$|\partial E|_K = ||\nu_{K'}||_{K,*} |\partial E|_{K'}, \quad \nu_K = \frac{\nu_{K'}}{||\nu_{K'}||_{K,*}}.$$

Indeed, for the closed unit disk $D \subset \mathbb{R}^2$ centered at 0 we know that in the Euclidean Lipschitz case $\nu_D = \nu_h$ and $|N_h| = ||N_h||_{D,*}$ where N is the *outer* unit normal. Hence we have

$$|\partial E|_K = ||\nu_h||_{K,*} d|\partial E|_D, \quad \nu_K = \frac{\nu_h}{||\nu_h||_{K,*}}.$$

Here $|\partial E|_D$ is the standard sub-Riemannian measure. Moreover, $\nu_h = N_h/|N_h|$ and $|N_h|^{-1} d|\partial E|_D = dS$, where dS is the standard Riemannian measure on S. Hence we get, for a set E with Euclidean Lipschitz boundary S

$$P_K(E, \Omega) = \int_{S\cap\Omega} ||N_h||_{K,*}\, dS, \tag{2.15}$$

where dS is the Riemannian measure on S, obtained from the area formula using a local Lipschitz parameterization of S, see Proposition 2.14 in [**36**]. It coincides with the 2-dimensional Hausdorff measure associated to the Riemannian distance induced by g. We stress that here N is the *outer* unit normal. This choice is important because of the lack of symmetry of $|| \cdot ||_K$ and $|| \cdot ||_{K,*}$.

Remark 2.5. If E has C^1 boundary ∂E, then its perimeter $P_K(E)$ is equal to the sub-Finsler area A_K of its boundary, defined by

$$A_K(\partial E) = \int_{\partial E} ||N_h||_{K,*} d\sigma. \tag{2.16}$$

where N_h is the projection on the horizontal distribution $\mathcal{H}$ of the Riemannian normal N with respect to the metric g, and $d\sigma$ is the Riemannian measure of ∂E. For more details see §2.4 in [**72**].

We will often omit the subscript K to simplify the notation.

3. The first variation of sub-Finsler area

3.1. First variation of sub-Finsler area. In this section we fix a convex body $K \subset \mathbb{R}^2$ containing 0 in its interior with C^2_+ boundary and consider its associated left-invariant norm $||\cdot||_K$ in $\mathbb{H}^1$. Since the convex body is fixed, we drop the subscript along this section.

Let S be an oriented C^2 surface immersed in $\mathbb{H}^1$. Let U be a C^2 vector field with compact support on S, normal component $u = \langle U, N\rangle$ and associated one-parameter group of diffeomorphisms $\{\varphi_s\}_{s\in\mathbb{R}}$. In this subsection we compute the first variation of the sub-Finsler area $A(s) = A(\varphi_s(S))$. More precisely

Theorem 3.1. *Let S be an oriented C^2 surface immersed in $\mathbb{H}^1$. Let U be a C^2 vector field with compact support on S, normal component $u = \langle U, N\rangle$ and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated one-parameter group of diffeomorphisms. Let $\eta = \pi(\nu_h)$. Then we have*

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S \big(u \operatorname{div}_S \eta - 2u\langle N, T\rangle\langle J(N_h), \eta\rangle\big)\, dS - \int_S \operatorname{div}_S \big(u\eta^\top\big)\, dS, \tag{3.1}$$

where div_S is the Riemannian divergence in S, and the superscript $\top$ indicates the tangent projection to S.

In the proof of Theorem 3.1 we shall make use of the following Lemma and its consequences.

LEMMA 3.2. *Let $\gamma : I \to \mathbb{H}^1$ be a C^1 curve, where $I \subset \mathbb{R}$ is an open interval, and V a horizontal vector field along γ. We have*

$$\frac{d}{ds}||V||_* = \langle \frac{D}{ds}V, \pi(V)\rangle + \langle \gamma', T_\gamma\rangle\langle V, J(\pi(V))\rangle. \tag{3.2}$$

PROOF. We fix $s_0 \in I$ and let $p = \gamma(s_0)$. Assume that $\pi(V(s_0)) = aX_p + bY_p$, for some $a, b \in \mathbb{R}$. Take the vector field $W(s) := aX_{\gamma(s)} + bY_{\gamma(s)}$ along γ. It coincides with $\pi(V(s_0))$ when $s = s_0$, and it is the restriction to γ of the left-invariant vector field $aX + bY$. In particular, $||(aX + bY)_{\gamma(s)}||_{\gamma(s)} = 1$ for all $s \in I$. Hence

$$||V(s)||_* \geqslant \langle V(s), (aX + bY)_{\gamma(s)}\rangle \quad \text{for all } s \in I,$$

and, since equality holds in the above inequality when $s = s_0$, we have

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=s_0} ||V(s)||_* &= \frac{d}{ds}\bigg|_{s=s_0} \langle V(s), (aX + bY)_{\gamma(s)}\rangle \\ &= \langle \frac{\nabla}{ds}\bigg|_{s=s_0} V(s), \pi(V(s_0))\rangle \end{aligned}$$

since

$$\frac{\nabla}{ds}\bigg|_{s=s_0} (aX + bY)_{\gamma(s)} = a\nabla_{\gamma'(s_0)}X + b\nabla_{\gamma'(s_0)}Y = 0.$$

The result follows from the relation between the covariant derivatives given in Equation (2.4). □

REMARK 3.3. In the proof of Lemma 3.2 we have obtained the equality

$$\frac{d}{ds}||V||_* = \langle \frac{\nabla}{ds}V, \pi(V)\rangle$$

for a horizontal vector field V along a curve γ. Since ∇ is a metric connection, we also have

$$\frac{d}{ds}||V||_* = \langle \frac{\nabla}{ds}V, \pi(V)\rangle + \langle V, \frac{\nabla}{ds}\pi(V)\rangle.$$

Hence we get

$$\langle V, \frac{\nabla}{ds}\pi(V)\rangle = 0 \tag{3.3}$$

for a horizontal vector field V along γ, where ∇/ds is the covariant derivative induced by the pseudo-hermitian connection on γ. Taking into account the relation between the Levi-Civita and pseudo-hermitian connections we deduce from (3.3) and (2.4)

$$\langle V, \frac{D}{ds}\pi(V) - \langle \dot{\gamma}, T_\gamma\rangle J(\pi(V))\rangle = 0. \tag{3.4}$$

The following is an easy consequence of Lemma 3.2

COROLLARY 3.4. *Let F be a vector field tangent to S and γ an integral curve of F. We have*

$$\langle \frac{D}{ds}\eta_\gamma, \nu_h\rangle = -\langle F, T\rangle\langle \eta, J(\nu_h)\rangle. \tag{3.5}$$

In particular, if F is horizontal,

$$\langle \frac{D}{ds}\eta_\gamma, \nu_h\rangle = 0. \tag{3.6}$$

PROOF. We take $V = \nu_h$ and we get (3.5) from equation (3.4). □

PROOF OF THEOREM 3.1. Standard variational arguments, see the proof of Lemma 4.3 in [**76**], yield

$$A'(0) = \frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S \left(\frac{d}{ds}\bigg|_{s=0} ||(N_s)_h||_* + ||N_h||_* \operatorname{div}_S U\right) dS,$$

where N_s is a smooth choice of unit normal to $\varphi_s(S)$ for small s. We fix a point $p \in S$ and consider the curve $\gamma(s) = \varphi_s(p)$. Lemma 3.2 now implies

$$\frac{d}{ds}\bigg|_{s=0} ||(N_s)_h||_* = \langle \frac{D}{ds}\bigg|_{s=0} (N_s)_h, \eta_p\rangle + \langle U_p, T_p\rangle\langle (N_h)_p, J(\eta_p)\rangle,$$

By the definition of $(N_s)_h$ we also have

$$\frac{D}{ds}\bigg|_{s=0} (N_s)_h = \frac{D}{ds}\bigg|_{s=0} \big(N_s - \langle N_s, T\rangle T\big),$$

where N_s is the Riemannian unit normal to $\varphi_s(S)$. A well-known lemma in Riemannian geometry implies

$$\frac{D}{ds}\bigg|_{s=0} N_s = -(\nabla_S u)(p) - A_S(U_p^\top),$$

where A_S is the Weingarten endomorphism of S. Since $\frac{D}{ds}\big|_{s=0} T = J(U_p)$ and η is horizontal, calling

$$B(U) = -\langle N, T\rangle\langle J(U), \eta\rangle + \langle U, T\rangle\langle N_h, J(\eta)\rangle,$$

we get

$$\begin{aligned} \tfrac{D}{ds}\big|_{s=0} ||(N_s)_h||_* &= \big(\langle -\nabla_S u - A_S(U^\top), \eta\rangle\big)_p + B(U_p) \\ &= -\langle \nabla_S u, \eta\rangle_p + B(U_p^\perp) + \big(-\langle A_S(U^\perp), \eta\rangle_p + B(U_p^\top)\big) \\ &= \big(-\langle \nabla_S u, \eta\rangle - 2u\langle N, T\rangle\langle J(N_h), \eta\rangle\big)_p + U_p^\top(||N_h||_*). \end{aligned}$$

Observe that

$$\begin{aligned} -\langle \nabla_S u, \eta\rangle &= u \operatorname{div}_S \eta - \operatorname{div}_S(u\eta) \\ &= u \operatorname{div}_S \eta - \operatorname{div}_S(u\eta^\top) - \operatorname{div}_S(u\langle N, \eta\rangle N) \\ &= u \operatorname{div}_S \eta - \operatorname{div}_S(u\eta^\top) - u||N_h||_* \operatorname{div}_S N. \end{aligned}$$

Hence we get

$$\begin{aligned} A'(0) = \int_S &\big(u \operatorname{div}_S \eta - 2u\langle N, T\rangle\langle J(N_h), \eta\rangle\big)\, dS \\ &+ \int_S \operatorname{div}_S \big(||N_h||_* U^\top - u\eta^\top\big)\, dS. \end{aligned}$$

From here we obtain formula (3.1) since the integral $\int_S ||N_h||_* U^\top dS$ is equal to 0 by the divergence theorem for Lipschitz vector fields. □

Now we simplify the first term appearing in the first variation formula (3.1).

LEMMA 3.5. *Let S be a C^2 surface immersed in $\mathbb{H}^1$ with unit normal N horizontal unit normal ν_h. Let $Z = J(\nu_h)$. Then we have*

$$\operatorname{div}_S \eta - 2\langle N, T\rangle\langle J(N_h), \eta\rangle = \langle D_Z \eta, Z\rangle. \tag{3.7}$$

PROOF. Let us consider the orthonormal basis in $S \setminus S_0$ given by the vector fields $Z = -J(\nu_h)$ and $E = \langle N, T\rangle \nu_h - |N_h| T = a\nu_h + bT$. Using equation (3.5) with $F = E$, we get

$$\begin{aligned}\langle D_E\eta, E\rangle &= a\langle D_E\eta, \nu_h\rangle + b\langle D_E\eta, T\rangle \\ &= -a\langle E, T\rangle\langle \eta, J(\nu_h)\rangle + b\big(E(\langle \eta, T\rangle) - \langle \eta, D_E T\rangle\big) \\ &= -ab\langle \eta, J(\nu_h)\rangle - ab\langle \eta, J(\nu_h)\rangle \\ &= -2ab\langle \eta, J(\nu_h)\rangle,\end{aligned}$$

as $D_E T = J(E) = aJ(\nu_h) = -aZ$. From $ab = -\langle N, T\rangle |N_h|$ we obtain

$$\langle D_E\eta, E\rangle = 2\langle N, T\rangle\langle \eta, J(N_h)\rangle.$$

Taking into account this equation and that $\operatorname{div}_S \eta = \langle D_Z\eta, S\rangle + \langle D_E\eta, E\rangle$, we obtain equation (3.7). □

DEFINITION 3.6. Given an oriented surface S immersed in $\mathbb{H}^1$ endowed with a smooth strictly convex left-invariant norm $||\cdot||_K$, its mean curvature is the function

$$H = \langle D_Z\eta_K, Z\rangle, \tag{3.8}$$

defined on $S \setminus S_0$.

REMARK 3.7. In [**79, 80**], the author obtained an expression of the mean curvature of a C^2 surface in terms of a parametrization when $\mathbb{H}^1$ is endowed with the left-invariant norm $\|\cdot\|_\infty$, and defined a notion of distributional mean curvature for polygonal norms.

COROLLARY 3.8. *Let S be an oriented C^2 surface immersed in $\mathbb{H}^1$. Let U be a C^2 vector field with compact support on $S \setminus S_0$, normal component $u = \langle U, N\rangle$ and associated one-parameter group of diffeomorphisms $\{\varphi_s\}_{s\in\mathbb{R}}$. Then*

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S uH\, dS, \tag{3.9}$$

where H is the mean curvature of S defined in (3.8).

By equation (3.8), a unit speed horizontal curve Γ contained in the regular part of a surface S satisfy the equation

$$\langle \frac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle = H, \tag{3.10}$$

where D/ds is the covariant derivative along Γ. Uniqueness of curves Γ satisfying (3.10) with given initial conditions $\Gamma(0), \dot\Gamma(0)$ cannot be obtained from (3.10). In the next result we prove that the horizontal components of Γ satisfy indeed an ordinary differential equation, thus providing uniqueness with given initial conditions.

COROLLARY 3.9. *Let S be a C^2 oriented surface immersed in $(\mathbb{H}^1, ||\cdot||)$ with mean curvature H. Let $\Gamma : I \to S \setminus S_0$ be a horizontal curve in the regular part of S parameterized by arc-length with $\Gamma(s) = (x_1(s), x_2(s), t(s))$. Then $\gamma(s) = (x_1, x_2)$ satisfies a differential equation of the form*

$$\ddot\gamma = F(\dot\gamma), \tag{3.11}$$

where $F(\dot\gamma) = H\,[A(\dot\gamma)](\dot\gamma)$ and A is a nonsingular C^1 matrix of order 2.

PROOF. Let $\frac{D}{ds}$ be the covariant derivative along the curve Γ. Since Γ is horizontal and parameterized by arc-length, the vector field $\frac{D}{ds}\dot\Gamma$ along Γ is proportional to $J(\dot\Gamma)$. Then there exists a function $\lambda : I \to \mathbb{R}$ such that

$$\tfrac{D}{ds}\dot\Gamma = \lambda J(\dot\Gamma).$$

Taking scalar product with $\eta = \pi(J(\dot\Gamma))$ we get

$$\lambda = \frac{\langle \frac{D}{ds}\dot\Gamma, \pi(J(\dot\Gamma))\rangle}{||J(\dot\Gamma)||_*} = \frac{\frac{d}{ds}\langle \dot\Gamma, \pi(J(\dot\Gamma))\rangle - H}{||J(\dot\Gamma)||_*}.$$

Hence we have

$$||J(\dot\Gamma)||_* \tfrac{D}{ds}\dot\Gamma - \dot f\, J(\dot\Gamma) = -H J(\dot\Gamma), \tag{3.12}$$

where $f = \langle \dot\Gamma, \pi(J(\dot\Gamma))\rangle$. Since $\dot\Gamma = \dot x_1 X + \dot x_2 Y$, $\frac{D}{ds}\dot\Gamma = \ddot x_1 X + \ddot x_2 Y$, and $J(\dot\Gamma) = -\dot x_2 X + \dot x_1 Y$, equation (3.12) is equivalent to the system

$$\begin{aligned} ||J(\dot\Gamma)||_*\, \ddot x_1 + \dot f \dot x_2 &= H \dot x_2, \\ ||J(\dot\Gamma)||_*\, \ddot x_2 - \dot f \dot x_1 &= -H \dot x_1. \end{aligned} \tag{3.13}$$

Let us compute $\dot f = df/ds$. Writing $\pi(aX + bY) = \pi_1(a,b)X + \pi_2(a,b)Y$ we have

$$f = \langle \dot\Gamma, \pi(J(\dot\Gamma))\rangle = \dot x_1 \pi_1(-\dot x_2, \dot x_1) + \dot x_2 \pi_2(-\dot x_2, \dot x_1)$$

and so:

$$\dot f = \left(\pi_1 + \dot x_1 \frac{\partial \pi_1}{\partial x_2} + \dot x_2 \frac{\partial \pi_2}{\partial x_2}\right)\ddot x_1 + \left(\pi_2 - \dot x_1 \frac{\partial \pi_1}{\partial x_1} - \dot x_2 \frac{\partial \pi_2}{\partial x_1}\right)\ddot x_2 = g\ddot x_1 + h\ddot x_2,$$

where the functions π_1, π_2 are evaluated at $(-\dot x_2, \dot x_1)$. Hence equation (3.13) is equivalent to

$$\begin{pmatrix} ||J(\dot\Gamma)||_* + g\dot x_1 & h\dot x_2 \\ -g\dot x_1 & ||J(\dot\Gamma)||_* - h\dot x_1 \end{pmatrix} \begin{pmatrix} \ddot x_1 \\ \ddot x_2 \end{pmatrix} = H \begin{pmatrix} \dot x_2 \\ -\dot x_1 \end{pmatrix} \tag{3.14}$$

The determinant of the square matrix in (3.14) is equal to

$$||J(\dot\Gamma)||_* \big(||J(\dot\Gamma)||_* + (g\dot x_1 - h\dot x_1)\big).$$

Since

$$g\dot x_1 - h\dot x_2 = \big(\pi_1 \dot x_2 - \pi_2 \dot x_1\big) + \sum_{i,j=1}^{2} \dot x_i \dot x_j \frac{\partial \pi_i}{\partial x_j} = -||J(\dot\Gamma)||_* + \sum_{i,j=1}^{2} \dot x_i \dot x_j \frac{\partial \pi_i}{\partial x_j}$$

we get that the determinant is equal to

$$||J(\dot\Gamma)||_* \sum_{i,j=1}^{2} \dot x_i \dot x_j \frac{\partial \pi_i}{\partial x_j}$$

and we write

$$\sum_{i,j=1}^{2} \dot x_i \dot x_j \frac{\partial \pi_i}{\partial x_j} = \begin{pmatrix} \dot x_1 & \dot x_2 \end{pmatrix} \begin{pmatrix} \partial\pi_1/\partial x_1 & \partial\pi_1/\partial x_2 \\ \partial\pi_2/\partial x_1 & \partial\pi_2/\partial x_2 \end{pmatrix} \begin{pmatrix} \dot x_1 \\ \dot x_2 \end{pmatrix}.$$

Since the kernel of $\big(\partial\pi_i/\partial x_j\big)_{ij}$ is generated by $(-\dot x_2, \dot x_1)$, we have

$$\begin{pmatrix} \partial\pi_1/\partial x_1 & \partial\pi_1/\partial x_2 \\ \partial\pi_2/\partial x_1 & \partial\pi_2/\partial x_2 \end{pmatrix} \begin{pmatrix} \dot x_1 \\ \dot x_2 \end{pmatrix} \neq 0,$$

and, since the image of $\big(\partial\pi_i/\partial x_j\big)_{ij}$ is generated by $(\dot x_1, \dot x_2)$, we get

$$\begin{pmatrix}\dot x_1 & \dot x_2\end{pmatrix}\begin{pmatrix}\partial\pi_1/\partial x_1 & \partial\pi_1/\partial x_2\\ \partial\pi_2/\partial x_1 & \partial\pi_2/\partial x_2\end{pmatrix}\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix}\neq 0.$$

So we can invert the matrix in (3.14) to get (3.11). □

REMARK 3.10. It is not difficult to prove that

$$\tfrac{D}{ds}\pi(J(\dot\Gamma)) = H\dot\Gamma - ||J(\dot\Gamma)||_*\,T.$$

Indeed it is only necessary to show that $\langle \frac{D}{ds}\pi(J(\dot\Gamma)), J(\dot\Gamma)\rangle = 0$, which follows from (3.6) using that $J(\dot\Gamma)=\nu_h$. Observe that the above equation is equivalent to

$$\big[\tfrac{D}{ds}\pi(J(\dot\Gamma))\big]_h = H\dot\Gamma.$$

Writing $\dot\Gamma = \dot x X + \dot y Y$, we have

$$\begin{pmatrix}\partial\pi_1/\partial x_1 & \partial\pi_1/\partial x_2\\ \partial\pi_2/\partial x_1 & \partial\pi_2/\partial x_2\end{pmatrix}\begin{pmatrix}-\ddot y\\ \ddot x\end{pmatrix} = H\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}.$$

However, since the determinant of the square matrix is 0 we cannot invert it to obtain an ordinary differential equation for $(\ddot x, \ddot y)$.

LEMMA 3.11. *Let $||\cdot||$ be a C^2_+ left-invariant norm in $\mathbb{H}^1$. Let $\gamma: I\to\mathbb{R}^2$ be a unit speed clockwise parameterization of a translation of the unit sphere of $||\cdot||$ in $\mathbb{R}^2$ by a vector $v\in\mathbb{R}^2$. Let Γ be a horizontal lifting of z. Then Γ satisfies the equation*

$$1 = \langle \tfrac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle. \tag{3.15}$$

PROOF. We have $\pi(J(\dot\Gamma)) = \pi_1(J(\dot\gamma))X + \pi_2(J(\dot\gamma))Y$. Since $J(\dot\gamma)$ is the outer normal to the unit sphere at $\gamma - v$ we have $\gamma - v = \big(\pi_1(J(\dot\Gamma)), \pi_2(J(\dot\Gamma))\big)$. Hence $\frac{D}{ds}\pi(J(\dot\Gamma)) = \dot x X + \dot y Y$ and we get (3.15). □

LEMMA 3.12. *Let $||\cdot||$ be a C^2_+ left-invariant norm in $\mathbb{H}^1$ and Γ a horizontal curve parameterized by arc-length satisfying the equation $\langle \frac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle = H$, with $H\in\mathbb{R}$. Then $\sigma(s) = h_\lambda(\Gamma(s/\lambda))$ is parameterized by arc-length and $\langle \frac{D}{ds}\pi(J(\dot\sigma)), \dot\sigma\rangle = H/\lambda$.*

PROOF. We have $\dot\sigma(s) = \dot\Gamma(s/\lambda)$ and $J(\dot\sigma(s)) = J(\dot\Gamma(s/\lambda))$. □

REMARK 3.13. Horizontal straight lines are solutions of

$$\langle \tfrac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle = 0$$

since $\dot\Gamma$ is the restriction of a left-invariant vector field in $\mathbb{H}^1$ and so they are $J(\dot\Gamma)$ and $\pi(J(\dot\Gamma))$.

THEOREM 3.14. *Let $||\cdot||$ be a C^2_+ left-invariant norm in $\mathbb{H}^1$. Let Γ be a horizontal curve satisfying the equation*

$$\langle \tfrac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle = H, \tag{3.16}$$

for some $H\geqslant 0$. Then Γ is either a horizontal straight line if $H=0$ or the horizontal lifting of a dilation and traslation of a unit speed clockwise parameterization of the circle $||\cdot|| = 1$ in $\mathbb{R}^2$ in case $H>0$.

PROOF. Horizontal straight lines and horizontal liftings of translations and dilations of the unit circle $||\cdot|| = 1$ in $\mathbb{R}^2$ satisfy equation (3.16). Uniqueness follow since the projection to $t = 0$ satisfy equation (3.11) and, by using translations and dilations, we can obtain any prescribed initial condition. $\square$

REMARK 3.15. The result in Theorem 3.14 includes that constant mean curvature surfaces for the sub-Riemannian area in the Heisenberg group are foliated by geodesics. This result can be found, with slight variations, in [**12, 15, 13, 42, 41**].

To finish this section we prove the following result, that holds trivially for variations supported in the regular part of S.

PROPOSITION 3.16. *Let S be a compact C^2 oriented surface in $(\mathbb{H}^1, ||\cdot||)$ enclosing a region E. Assume that S has constant mean curvature H and a finite number of singular points. Then*

(1) *S is a critical point of the sub-Finsler area for any volume-preserving variation.*
(2) *S is a critical point of the functional $A - H\,|\cdot|$.*

PROOF. It is only necessary to prove that if U is a smooth vector field with compact support in $\mathbb{H}^1$ and $\{\varphi_s\}_{s\in\mathbb{R}}$ is its associated flow, then

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S H u\, dS.$$

From formula (3.1) this is equivalent to proving that

$$\int_S \operatorname{div}_S \left(u\eta^\top\right) dS = 0.$$

To compute the integral $\int_S u\eta^\top dS$ we consider the finite number of singular points $p_1, \ldots, p_n$, and take small disjoint balls $B_i(p_i)$ centered at the points p_i. For $\varepsilon > 0$ small enough so that the balls $B_\varepsilon(p_i)$ are contained in B_i we have

$$\int_{S\setminus\bigcup_{i=1}^n B_\varepsilon(p_i)} \operatorname{div} u\eta^\top\, dS = \sum_{i=1}^n \int_{\partial B_\varepsilon(p_i)} \langle \xi_i, u\eta^\top\rangle\, d(\partial B_\varepsilon(p_i)),$$

where ξ_i is the unit inner normal to $\partial B_\varepsilon(p_i)$. Since $u\eta^\top$ is bounded and the lengths of $\partial B_\varepsilon(p_i)$ go to 0 when $\varepsilon \to 0$ we have

$$\lim_{\varepsilon\to 0} \sum_{i=1}^n \int_{\partial B_\varepsilon(p_i)} \langle \xi_i, u\eta^\top\rangle\, d(\partial B_\varepsilon(p_i)) = 0.$$

Since the modulus of

$$\begin{aligned}\operatorname{div}_S(u\eta^\top) &= \langle \nabla_S u, \eta^\top\rangle + u \operatorname{div}_S \eta^\top \\ &= \langle \nabla_S u, \eta^\top\rangle + u\left(\operatorname{div}_S \eta - \langle \eta^\top, N\rangle \operatorname{div}_S N\right)\end{aligned}$$

is uniformly bounded, the dominated convergence theorem implies

$$\begin{aligned}\int_S \operatorname{div}_S u\eta^\top dS &= \lim_{\varepsilon\to 0} \int_{S\setminus\bigcup_{i=1}^n B_\varepsilon(p_i)} \operatorname{div} u\eta^\top\, dS \\ &= \lim_{\varepsilon\to 0} \sum_{i=1}^n \int_{\partial B_\varepsilon(p_i)} \langle \xi_i, u\eta^\top\rangle\, d(\partial B_\varepsilon(p_i)) = 0.\end{aligned}$$

$\square$

COROLLARY 3.17 (Minkowski formula). *Let S be a compact C^2 oriented surface in $(\mathbb{H}^1, ||\cdot||)$ enclosing a region E. Assume that S has constant mean curvature H and a finite number of singular points. Then*

$$3A(S) - 4H|E| = 0. \tag{3.17}$$

PROOF. We consider the vector field $W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial t}$ and its associated flow $\varphi_s((x,y,t)) = (e^s x, e^s y, e^{2s} t)$. Since

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = 3A(S), \quad \frac{d}{ds}\bigg|_{s=0} |\varphi_s(E)| = 4|E|,$$

Proposition 3.16 implies

$$0 = \frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) - H\frac{d}{ds}\bigg|_{s=0} |\varphi_s(E)| = 3A(S) - 4H|E|. \qquad \square$$

3.2. The mean curvature of a horizontal graph. Assume that Ω is an open set of the xy plane and that $u : \Omega \to \mathbb{R}$ is a C^1 function. The sub-Finsler area A_K of the graph of u, when K is a convex body of class C^2_+, can be computed as

$$A_K(\text{graph}(u)) = \int_\Omega ||\nabla u + F||_* d\mathcal{L}^2 \tag{3.18}$$

from equation (2.16), since the horizontal unit normal N_h in the graph is given by

$$N_h = \frac{(u_x - y, u_y + x)}{(1 + (u_x - y)^2 + (u_y + x)^2)^{1/2}},$$

and

$$dS = (1 + (u_x - y)^2 + (u_y + x)^2)^{1/2}\, d\mathcal{L}^2.$$

We remark that the area in equation (3.18) is computed with respect to the *downward* pointing unit normal to the graph (the one with negative third coordinate). Since the norm $||\cdot||_K$ is asymmetric, taking the opposite normal would give a different area.

For the mean curvature of the graph of a C^2 function we have the following result.

THEOREM 3.18. *Let K be a convex body of class C^2_+ with $0 \in \text{int}(K)$. Let Ω be an open set in the xy plane, and $u : \Omega \to \mathbb{R}$ a C^2 function. Then the sub-Finsler mean curvature H_K in the regular part of the graph of u with respect to the downward pointing unit normal is given by*

$$H_K = \text{div}\big(\pi_K(\nabla u + F)\big). \tag{3.19}$$

PROOF. Let $v : \Omega \to \mathbb{R}$ be a C^∞ function with compact support in the projection of the regular set of graph(u) to Ω. We consider a vector field U with compact support which coincides with the vector field vT near the graph of u. If φ_s is the one parameter group of diffeomorphisms associated to U then $\varphi_s(x, y, u(x,y)) = (x, y, u(x,y) + sv(x,y))$ for all $(x,y) \in \Omega$ and s small enough. Hence $\varphi_s(\text{graph}(u)) = \text{graph}(u + sv)$ for s small enough. Since $||\cdot||_K$ is of class C^2_+ we can represent the dual norm in terms of π_K to compute the first variation. So we have

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0} A_K(\varphi_s(\mathrm{graph}(u)) &= \int_\Omega \frac{d}{ds}\bigg|_{s=0} \langle\nabla(u+sv)+F, \pi_K(\nabla(u+sv)+F)\rangle\, d\mathcal{L}^2 \\ &= \int_\Omega \langle \nabla v, \pi_K(\nabla u+F)\rangle\, d\mathcal{L}^2 \\ &\quad + \int_\Omega \langle \nabla u+F, \frac{d}{ds}\bigg|_{s=0} \pi_K(\nabla(u+sv)+F)\rangle\, d\mathcal{L}^2 \\ &= -\int_\Omega v \ \mathrm{div}\,\big(\pi_K(\nabla u+F)\big)\, d\mathcal{L}^2 \\ &= \int_{\mathrm{graph}(u)} \langle U,N\rangle \ \mathrm{div}\,\big(\pi_K(\nabla u+F)\big)\, dS.\end{aligned}$$

In the third equality we have used equation (3.3) and in the fourth one that $-v\, d\mathcal{L}^2 = \langle U,N\rangle\, dS$. Comparing the last formula with the first variation formula involving the mean curvature (3.9) we obtain (3.19). □

Remark 3.19. Recall that in the sub-Riemannian case K is the unit disk D and that $\pi_D(v) = v/|v|$ for any $v \neq 0$, so the *sub-Riemannian* mean curvature of the graph satisfies

$$H_D = \mathrm{div}\left(\frac{\nabla u+F}{|\nabla u+F|}\right),$$

see Cheng et al. [**14**].

3.3. The singular set of C^2 surfaces with constant mean curvature. The singular set of a C^2 surface with a bounded condition on the sub-Riemannian mean curvature was considered in a paper by Cheng et al. [**12**] and, for symmetric sub-Finsler norms, in § 7 of [**34**]. In both cases, which include the case where the mean curvature is constant, it is composed of isolated points and C^1 curves. In the general sub-Finsler case we have the following result.

Theorem 3.20. *Let $K \subset \mathbb{R}^2$ be a convex body with C^2_+ boundary and $0 \in \mathrm{int}(K)$, and let $S \subset \mathbb{H}^1$ be a C^2 surface with constant mean curvature. Then the singular set S_0 of S is composed of isolated points and C^1 curves.*

Proof. We take $p_0 = (x_0, y_0, t_0) \in S_0$. Since $T_{p_0}S = \mathcal{H}_{p_0}$ we can describe the surface around p_0 as the graph of a C^2 function $u : \Omega \to \mathbb{R}$, where Ω is an open set in the xy plane containing the projection (x_0, y_0) of p_0. The graph of u, graph(u), is an open set of S containing p_0.

The intersection graph$(u) \cap S_0$ is composed of the points of graph(u) whose projection (x, y) satisfies the equations

$$u_x - y = u_y + x = 0. \tag{3.20}$$

Let us consider the map $G : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G = (u_x - y, u_y + x).$$

Its derivative is given by

$$dG = \begin{pmatrix} u_{xx} & u_{xy}-1 \\ u_{xy}+1 & u_{yy}\end{pmatrix}.$$

We observe that rank $dG \geqslant 1$ since $u_{xy}-1$, $u_{xy}+1$ cannot vanish simultaneously.

If $\operatorname{rank} dG(x_0, y_0) = 2$ then the inverse function theorem implies that there is a neighborhood $U \subset \Omega$ of (x_0, y_0) so that (x_0, y_0) is the only point in U which satisfies equation (3.20). Hence p_0 is an isolated singular point of S.[1]

If $\operatorname{rank} dG(x_0, y_0) = 1$ then the kernel of $dG(x_0, y_0)$ is 1-dimensional. We take any $(a, b) \in \mathbb{R}^2$ with $a^2 + b^2 = 1$ and such that

$$(a, b) \cdot dG(x_0, y_0) \neq (0, 0).$$

The gradient of the function $F_{a,b} : \Omega \to \mathbb{R}$, defined by $F_{a,b} = a\,(u_x - y) + b\,(u_y + x)$, is equal to $(a, b) \cdot dG(x, y)$. Since $\nabla F_{a,b}(x_0, y_0) \neq (0, 0)$, the implicit function theorem implies the existence of an open set $U_{a,b} \subset \Omega$ so that

$$\big\{(x, y) \in U_{a,b} : F_{a,b}(x, y) = 0\big\}$$

is a C^1 curve.

Now fix another $(a', b') \in \mathbb{R}^2$ such that $(a')^2 + (b')^2 = 1$, $(a', b') \cdot dG(x_0, y_0) \neq (0, 0)$, and so that (a, b) and (a', b') are linearly independent. Taking $U = U_{a,b} \cap U_{a',b'}$ we conclude that

$$\big\{(x, y) \in U : F_{a,b}(x, y) = 0\big\}, \qquad \big\{(x, y) \in U : F_{a',b'}(x, y) = 0\big\}$$

are C^1 curves $\Gamma_{a,b}$ and $\Gamma_{a',b'}$, respectively. The projection of $S_0 \cap \operatorname{graph}(u)$ inside U is contained in $\Gamma_{a,b} \cap \Gamma_{a',b'}$. Moreover, since $(a, b), (a', b')$ are linearly independent, we conclude that the projection of $S_0 \cap \operatorname{graph}(u)$ inside U is *exactly* the set $\Gamma_{a,b} \cap \Gamma_{a',b'}$.

To end the proof let us show that the C^1 curves $\Gamma_{a,b}$, $\Gamma_{a',b'}$ coincide in an open neighborhood of (x_0, y_0). Observe that $\Gamma_{a,b}$ and $\Gamma_{a',b'}$ are tangent at (x_0, y_0). We reason by contradiction assuming that $\Gamma_{a,b}$ and $\Gamma_{a',b'}$ do not coincide in any neighborhood of (x_0, y_0).

Assume first that p_0 is not an isolated point of S_0. Then there is a sequence of piecewise smooth domains Ω_i converging in Hausdorff distance to (x_0, y_0). Each one is bounded by a segment of $\Gamma_{a,b}$ and a segment of $\Gamma_{a',b'}$ meeting at the endpoints. Observe that (a, b) is perpendicular to $\nabla u + F$ on $\Gamma_{a,b}$ and (a', b') is perpendicular to $\nabla u + F$ on $\Gamma_{a',b'}$. Hence $\pi_K(\nabla u + F)$ is a constant vector on each component, equal to $\pi_K(\pm(-b, a))$ on $\Gamma_{a,b}$ and to $\pi_K(\pm(-b', a'))$ on $\Gamma_{a',b'}$. On the other hand, the outer unit normal ν_i to Ω_i approaches the unit vector $\pm\nabla F_{a,b}/|\nabla F_{a,b}|(x_0, y_0)$ so that $\langle \pi_K(\nabla u + F), \nu_i\rangle$ approaches the quantity $c_1 = \pm\langle \pi_K(\pm(-b, a)), \nu_0\rangle$ on $\Gamma_{a,b}$ and $c_2 = \pm\langle \pi_K(\pm(-b', a'), \nu_0\rangle$. For i large enough, the distance function r to (x_0, y_0) is monotone over the curves $\Gamma_{a,b}$ and $\Gamma_{a',b'}$ and we have

$$0 < r_i \leqslant r(x, y) \leqslant s_i$$

for any $(x, y) \in \Omega_i$. Finally observe that Ω_i is contained in a cone of vertex (x_0, y_0) and angle θ_i, with $\lim_{i\to\infty} \theta_i = 0$ since $\Gamma_{a,b}, \Gamma_{a',b'}$ are tangent at (x_0, y_0).

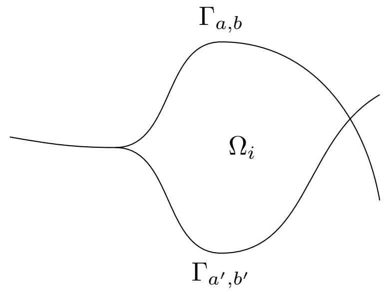
Applying the divergence theorem to the mean curvature equation (3.19) we get

$$\int_{\partial\Omega_i} \langle \pi_K(\nabla u + F), \nu_i\rangle = \int_{\Omega_i} \operatorname{div}(\pi_K(\nabla u + F)) = \int_{\Omega_i} H.$$

We estimate

$$\bigg| \int_{\partial\Omega_i} \langle \pi(\nabla u + F), \nu_i\rangle \bigg| \geqslant \frac{|c_1 \pm c_2|}{2}\,(s_i - r_i).$$

[1]Moreover, Lemma 3.2 in [12] implies that, if p_0 is not isolated in S_0 then det $dG(x_0, y_0) = 0$ and so $\operatorname{rank} dG(x_0, y_0) = 1$.



On the other hand, assuming $s_i \leqslant 1$ we get

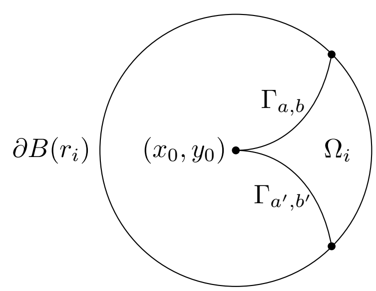
$$\left| \int_{\Omega_i} H \right| \leqslant \left| \int_{r_i}^{s_i} \int_{\alpha_i}^{\alpha_i+\theta_i} H \, r dr d\theta \right| \leqslant |H| \theta_i (s_i - r_i).$$

Hence we get

$$\frac{|c_1 \pm c_2|}{2} \leqslant |H| \theta_i.$$

Letting $\theta_i \to 0$ we get $c_1 = \pm c_2$, a contradiction since (a, b) and (a', b') are linearly independent.

Finally we consider the case when $\Gamma_{a,b}$ and $\Gamma_{a',b'}$ do not meet in a neighborhood of (x_0, y_0) except at (x_0, y_0). In this case we take the region Ω_i surrounded by two segments of $\Gamma_{a,b}$, $\Gamma_{a',b'}$ leaving (x_0, y_0) in the same direction, and $\partial B(r_i)$, where the Euclidean ball $B(r_i) \subset \mathbb{R}^2$ is centered at (x_0, y_0).



Since the arc-length of $\overline{\Omega}_i \cap \partial B(r_i)$ is bounded by $\theta_i r_i$, we conclude that $c_1 = \pm c_2$ as in the previous case. □

Remark 3.21. The hypothesis on the mean curvature in Theorem 3.3 in [**12**] is that $H \leqslant C/r$ near a singular point for some constant $C > 0$. Here r is the Euclidean distance to the singular point. In this case, using the notation in the proof of Theorem 3.20, it is enough to estimate

$$\left| \int_{\Omega_i} H \right| \leqslant \left| \int_{r_i}^{s_i} \int_{\alpha_i}^{\alpha_i+\theta_i} \frac{C}{r} \, r dr d\theta \right| \leqslant C \theta_i (s_i - r_i).$$

3.4. Existence of isoperimetric sets. For the existence of isoperimetric sets we follow the paper by Leonardi and Rigot [**56**] on Carnot groups, and the adaptation to symmetric sub-Finsler norms in § 3 of [**34**].

3.5. Pansu-Wulff spheres and examples. We consider a convex body $K \subset \mathbb{R}^2$ containing 0 in its interior and the associated norm $||\cdot||_K$ in $\mathbb{H}^1$.

Definition 3.22. Consider a clockwise-oriented L-periodic parameterization $\gamma : \mathbb{R} \to \mathbb{R}^2$ of the curve $||\cdot||_K = 1$. For fixed $v \in \mathbb{R}$ take the translated curve $u \mapsto \gamma(u+v) - \gamma(v)$ and its horizontal lifting $\Gamma_v : \mathbb{R} \to \mathbb{H}^1$ with initial point $(0,0,0)$ at $u = 0$.

The set $\mathbb{S}_K$ is defined as

$$\mathbb{S}_K = \bigcup_{v\in[0,L)} \Gamma_v([0,L]). \tag{3.21}$$

We shall refer to $\mathbb{S}_K$ as the *Pansu-Wulff sphere* associated to the left-invariant norm $||\cdot||_K$.

When $K = D$, the closed unit disk centered at the origin in $\mathbb{R}^2$, the Pansu-Wulff sphere $\mathbb{S}_D$ is Pansu's sphere, see [**66, 67**].

Remark 3.23. In the construction of the Pansu-Wulff sphere we are not assuming any regularity on the boundary of K. Since ∂K is a locally Lipschitz curve, its horizontal lifting is well defined.

Remark 3.24. The set $\mathbb{S}_K$ is union of curves leaving from $(0,0,0)$ that meet again at the point $(0,0,2|K|)$. Since γ is L-periodic, the construction is L-periodic in v and so $\mathbb{S}_K$ is the image of a continuous map from a sphere to $\mathbb{H}^1$.

Example 3.25. Given the Euclidean norm $|\cdot|$ in $\mathbb{R}^2$ and $a = (a_1, a_2)$, where $a_1, a_2 > 0$, we define the norm:

$$||(x_1, x_2)||_a = |(\tfrac{x_1}{a_1}, \tfrac{x_2}{a_2})|.$$

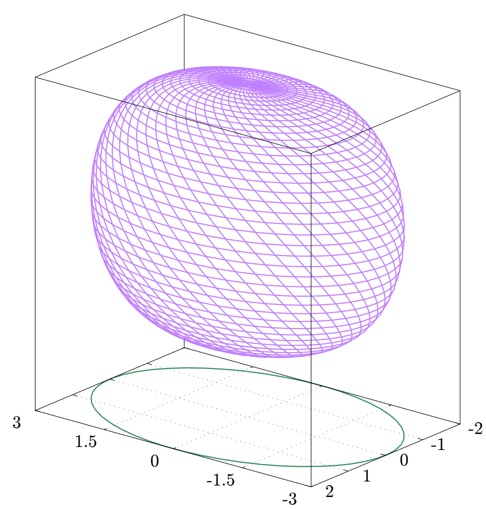


Figure 3. The Pansu-Wulff sphere associated to the norm $||\cdot||_a$ with $a = (1, 1.5)$. Observe that the projection to the horizontal plane $t = 0$ is an ellipse with semiaxes of lengths 2 and 3.

The unit ball K_a for this norm is an ellipsoid with axes of length a_1 and a_2. We parameterize clockwise the unit circle of the norm $||\cdot||_K$ by

$$\gamma(s) = (a_1 \sin(s), a_2 \cos(s)), \quad s \in \mathbb{R}.$$

This parameterization is injective of period 2π. The translation of this curve to the origin by the point $-\gamma(v)$ is given by the curve

$$\Lambda_v(u) = \gamma(u+v) - \gamma(v).$$

The horizontal lifting of Λ_v is given by $(\Lambda_v(u), t_v(u))$, where

$$t_v(u) = \int_0^u \big[\Lambda_v(\xi)\cdot J(\dot\Lambda_v(\xi))\big]\,d\xi.$$

Since

$$\Lambda_v(\xi)\cdot J(\dot\Lambda_v(\xi)) = (\gamma(\xi+v)-\gamma(v))\cdot J(\dot\gamma(\xi+v)),$$

we get

$$t_v(u) = a_1a_2\big(u + \sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big).$$

Hence a parameterization of $\mathbb{S}_{K_a}$ is given by

$$\begin{aligned} x(u,v) &= a_1\big(\sin(u+v) - \sin(v)\big) \\ y(u,v) &= a_2\big(\cos(u+v) - \sin(v)\big), \\ t(u,v) &= a_1a_2\big(u + \sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big). \end{aligned}$$

Example 3.26. Given any convex set K containing 0 in its interior, we can parameterize its Lipschitz boundary ∂K as

$$\gamma(s) = \big(x(s), y(s)\big) = r(s)\,\big(\sin(s), \cos(s)\big), \quad s \in \mathbb{R}.$$

where $r(s) = \rho(\sin(s), \cos(s))$ and ρ is the radial function of K defined as $\rho(u) = \sup\{\lambda \geqslant 0 : \lambda u \in K\}$ for any vector u of modulus 1 in $\mathbb{R}^2$.

A horizontal lifting of the curve γ passing through the point $(\gamma(0), 0)$ can be obtained computing

$$t(s) = \int_0^s \gamma(\xi)\cdot J(\dot\gamma(\xi))\,d\xi = \int_0^s r^2(\xi)\,d\xi,$$

since $J(\dot\gamma(s)) = r(s)\,(\sin(s), \cos(s)) + \dot r(s)\,(-\cos(s), \sin(s))$. Hence the curve

$$\Gamma(s) = \big(x(s), y(s), t(s)\big) = \Big(\gamma(s), \int_0^s r^2(\xi)\,d\xi\Big)$$

is a horizontal lifting of the curve γ.

Now we translate all these curves to pass through the origin of $\mathbb{H}^1$. This way we get the parameterization Φ_K of $\mathbb{S}_K$ given by

$$(u,v) \mapsto \ell_{-\Gamma(v)}(\Gamma(u+v))$$

for $(u,v) \in [0,2\pi]^2$. Since

$$\ell_{(x_0,y_0,t_0)}(x,y,t) = \big(x+x_0, y+y_0, t+t_0+(xy_0-x_0y)\big),$$

computing the left-translation using the expression for Γ obtained before we get

$$\begin{aligned} x(u,v) &= r(u+v)\sin(u+v) - r(v)\sin(v), \\ y(u,v) &= r(u+v)\cos(u+v) - r(v)\cos(v), \\ t(u,v) &= r(v)r(u+v)\big(\sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big) \\ &\quad + \textstyle\int_v^{u+v} r^2(\xi)\,d\xi. \end{aligned} \tag{3.22}$$

The parameterization given by equations (3.22) is useful to obtain regularity properties of $\mathbb{S}_K$. If ∂K is of class C^ℓ, $\ell \geqslant 0$, its radial function $r(s) = (x(s)^2 + y(s)^2)^{1/2}$ is of class C^ℓ and hence the parameterization Φ_K is an immersion of class C^ℓ for $0 < u < 2\pi$.

EXAMPLE 3.27. Let $\ell > 1$. We consider the ℓ-norm in $\mathbb{R}^2$ defined as

$$||(x_1, x_2)||_\ell = \left(|x_1|^\ell + |x_2|^\ell\right)^{1/\ell}.$$

Denote by K_ℓ the unit ball for this ℓ-norm. We can parametrize the unit circle $||\cdot||_\ell = 1$ using (3.22). In this case

$$\rho(x, y) = \frac{1}{\left(|x|^\ell + |y|^\ell\right)^{1/\ell}}, \qquad |(x, y)| = 1.$$

By the previous example, the Pansu-Wulff sphere $\mathbb{S}_{K_\ell}$ is parameterized by equations (3.22).

REMARK 3.28. Assume we have a sequence of of convex sets (K_i) converging in Hausdorff distance to a limit convex set K. Then the radial functions r_{K_i} uniformly converge to the radial function r of the limit set K. Hence equations (3.22) imply that the Pansu-Wulff spheres $\mathbb{S}_{K_i}$ converge in Hausdorff distance to a ball bounded by the horizontal liftings of translations of a parameterization γ of ∂K.

Since $\lim_{\ell\to 1} ||\cdot||_\ell = ||\cdot||_1$ and $\lim_{\ell\to\infty} ||\cdot||_\ell = ||\cdot||_\infty$, we can use the previous argument to show that the Pansu-Wulff spheres $\mathbb{S}_{K_\ell}$ converge to the two spheres $\mathbb{S}_1$ and $\mathbb{S}_\infty$. Under these conditions, it is not difficult to check that the corresponding perimeters converge to the limit perimeter.

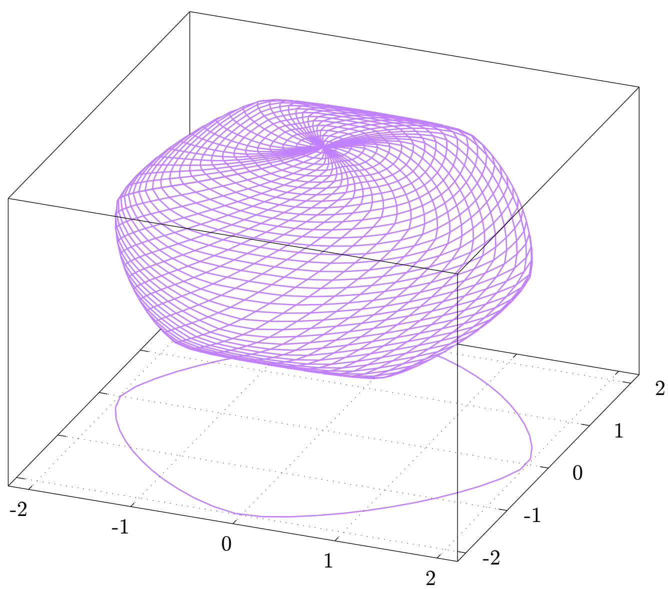


FIGURE 4. The Pansu-Wulff sphere $\mathbb{S}_{K_\ell}$ for the ℓ-norm, $\ell = 1.5$. The horizontal curve is the projection of the equator to the plane $t = 0$. We observe that the Pansu-Wulff sphere projects to the set $||\cdot||_\ell \leqslant 2$ in the $t = 0$ plane.

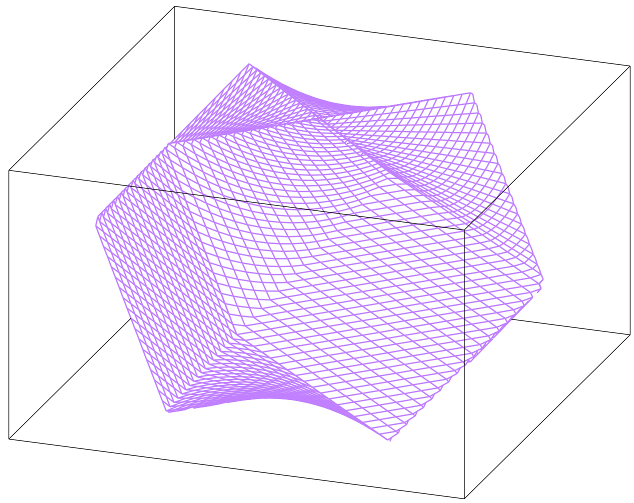


FIGURE 5. The sphere $\mathbb{S}_1$ obtained as Hausdorff limit of the Pansu-Wulff spheres $\mathbb{S}_{K_r}$ of the ℓ-norm when ℓ converges to 1

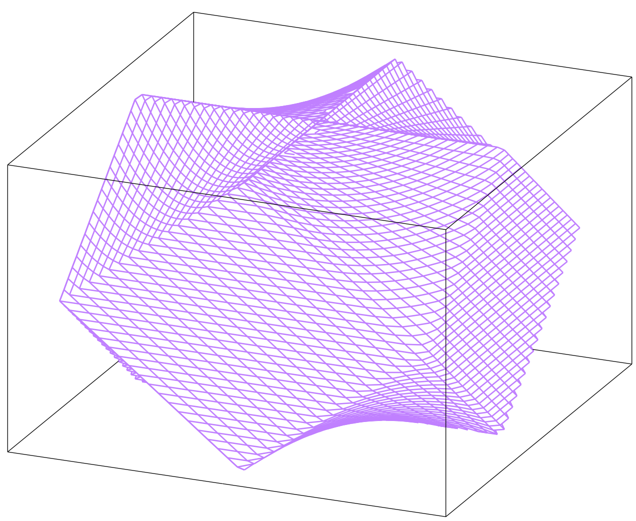


FIGURE 6. The sphere $\mathbb{S}_\infty$ obtained as Hausdorff limit of the Pansu-Wulff spheres $\mathbb{S}_{K_r}$ of the ℓ-norm when ℓ converges to ∞

EXAMPLE 3.29. Let us consider the equilateral triangle T in the plane $\mathbb{R}^2$ defined as the convex envelope of the points $a_1 = (0,1)$, $a_2 = (\sqrt{3}/2, -1/2)$, $a_3 = (-\sqrt{3}/1, -1/2)$. We can define a norm $||\cdot||_T$ by the equality

$$||x||_T = \max\big\{\langle x, a_i\rangle : i = 1,2,3\big\}, \quad x \in \mathbb{R}^2.$$

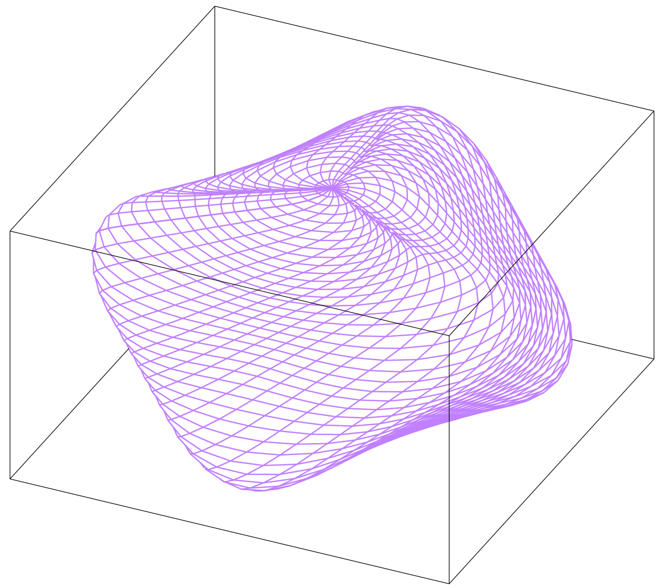


FIGURE 7. The Pansu-Wulff sphere $\mathbb{S}_{T,\ell}$ for the norm $||\cdot||_{T,\ell}$, with $r = 2$.

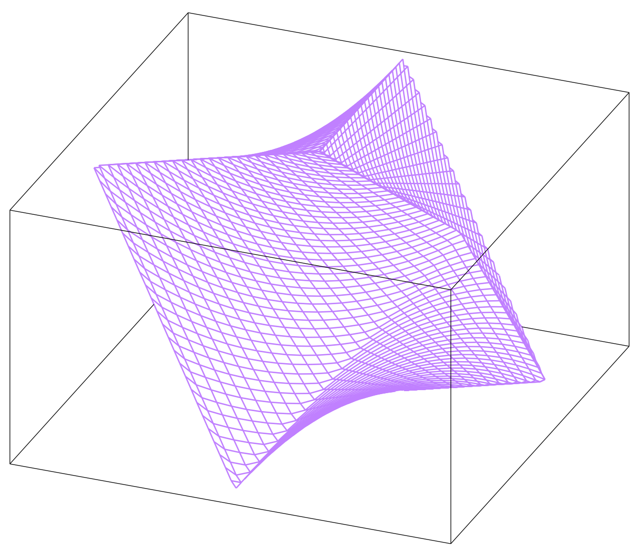


FIGURE 8. The sphere $\mathbb{S}_T$ obtained as limit of the Pansu-Wulff spheres $\mathbb{S}_{T,\ell}$ when $r \to \infty$.

The unit ball of the norm $||\cdot||_T$ is the triangle T. It is neither smooth nor strictly convex. However we may consider the approximating norms

$$||x||_{T,\ell} = \left(\sum_{i=1}^{3} \max\{\langle x, a_i\rangle, 0\}^{\ell}\right)^{1/\ell}.$$

These norms are smooth and strictly convex and $\lim_{\ell\to\infty} ||x||_{T,\ell} = ||x||_T$. Hence the Pansu-Wulff spheres $\mathbb{S}_{K_{T,\ell}}$ converge in Hausdorff distance when $\ell \to \infty$ to the sphere $\mathbb{S}_T$ obtained by traslating ∂T to touch the origin and lifting the obtained curves as horizontal ones to $\mathbb{H}^1$.

3.6. Geometric properties of the Pansu-Wulff spheres. In this section we show several geometric properties of the Pansu-Wulff spheres $\mathbb{S}_K$ associated with a left-invariant norm $||\cdot||_K$. We start by looking at the projection of the sphere to the $t = 0$ plane. This projection is determined by the geometry of the convex set K.

Given a convex body $K \subset \mathbb{R}^n$, the *difference body* of K is the set

$$\mathrm{D}K = K - K = \{x - y : x, y \in K\}.$$

The difference body $\mathrm{D}K$ is a *centrally symmetric* convex body. This means that $-x \in \mathrm{D}K$ whenever $x \in \mathrm{D}K$. If h_K is the support function of K then the support function of $\mathrm{D}K$ is given by

$$h_{\mathrm{D}K}(u) = h_K(u) + h_K(-u),$$

see [**78**, p. 140]. This is the width of K in the direction of u.

LEMMA 3.30. *Let $K \subset \mathbb{R}^n$ be a convex body with $0 \in \mathrm{int}(K)$. We consider the set*

$$K_0 = \bigcup_{p\in\partial K} (-p + K). \tag{3.23}$$

Then we have

1. *$0 \in K_0$.*
2. *K_0 is a convex body.*
3. *K_0 is the difference body of K. In particular, K_0 is centrally symmetric.*
4. *If K is centrally symmetric then $K_0 = 2K$.*
5. *We have*
$$\bigcup_{p\in\partial K} (-p + K) = \bigcup_{p\in\partial K} (-p + \partial K).$$

PROOF. To prove 1 take into account that $0 = -p + p \in -p + K \subset K_0$ for any $p \in \partial K$.

To prove 2, we take $p_1, p_2 \in \partial K$, $q_1, q_2 \in K$ and $\lambda \in [0, 1]$. Then

$$\lambda(-p_1 + q_1) + (1 - \lambda)(-p_2 + q_2) = -p_\lambda + q_\lambda,$$

where

$$p_\lambda = \lambda p_1 + (1 - \lambda)p_2, \qquad q_\lambda = \lambda q_1 + (1 - \lambda)q_2.$$

If $p_\lambda = q_\lambda$ then $-p_\lambda + q_\lambda = 0 \in K_0$. Otherwise the segment $[p_\lambda, q_\lambda]$ is not trivial and contained in K. Let $\mu_0 \geqslant 1$ such that $q_\lambda + \mu_0(p_\lambda - q_\lambda) \in \partial K$. The value μ_0 is computed as the supremum of the set $\{\mu \geqslant 0 : q_\lambda + \mu(p_\lambda - q_\lambda) \in K\}$. We have

$$-p_\lambda + q_\lambda = -(q_\lambda + \mu_0(p_\lambda - q_\lambda)) + (q_\lambda + (\mu_0 - 1)(p_\lambda - q_\lambda)).$$

The point $q_\lambda + \mu_0(p_\lambda - q_\lambda)$ belongs to ∂K by the choice of μ_0 and the point $q_\lambda + (\mu_0 - 1)(p_\lambda - q_\lambda)$ belongs to K since $0 \leqslant \mu_0 - 1 \leqslant \mu_0$. Hence $-p_\lambda + q_\lambda \in K_0$ and so K_0 is convex.

To prove 3, we take a vector v with $\langle v, v\rangle = 1$. Let $q \in \partial K_0$ such that

$$h_{K_0}(v) = \langle q, v\rangle \geqslant \langle z, v\rangle \quad \forall\ z \in K_0. \tag{3.24}$$

By the definition of K_0, there exists $p \in \partial K$ such that $q \in -p+K$. We claim that $q \in -p+\partial K$: otherwise $p+q \in \operatorname{int}(K)$ and there exists $\varepsilon > 0$ such that $p+q+\varepsilon v \in K$. So we have

$$\langle -p+(p+q+\varepsilon v), v\rangle = \langle q+\varepsilon v, v\rangle = \langle q, v\rangle + \varepsilon > \langle q, v\rangle.$$

Since $p+q+\varepsilon v \in K$ this yields a contradiction. Hence $q \in -p+\partial K = \partial(-p+K)$ for some $p \in \partial K$.

Since $-p+K \subset K_0$ and q is a boundary point for both sets, we deduce that v is a normal vector to $-p+K$ at q. As $h_{-p+K}(v) = -\langle p, v\rangle + h_K(v)$,we have

$$h_{K_0}(v) = h_{-p+K}(v) = h_K(v) + \langle p, -v\rangle.$$

It remains to prove that $h_K(-v) = \langle p, -v\rangle$. Assume by contradiction that $\langle p, -v\rangle < h_K(-v) = \langle x, -v\rangle$ for some $x \in \partial K$. Then we have

$$\langle -x+(p+q), v\rangle = \langle -x+p, v\rangle + \langle q, v\rangle > \langle q, v\rangle,$$

that cannot hold by (3.24) since $p+q \in K$ and so $-x+p+q \in -x+K \subset K_0$.

To prove 4, we note that $h_K(v) = h_K(-v)$ when K is centrally symmmetric and, by 3, $h_{K_0} = 2h_K$. Hence $K = 2K_0$.

Finally, to prove 5 we notice that $\bigcup_{p\in\partial K}(-p+K) \supset \bigcup_{p\in\partial K}(-p+\partial K)$. To prove the remaining inclusion we take $p \in \partial K$ and $u \in K$ such that $q = -p+u \in \bigcup_{p\in\partial K}(-p+K)$. Then Lemma 3.31 allows us to find $p_1, u_1 \in \partial K$ such that $q = -p+u = -p_1+u_1$. Hence $q \in \bigcup_{p\in\partial K}(-p+\partial K)$. □

Lemma 3.31. *Let $K \subset \mathbb{R}$ be a convex body, and $a, b \in K$. Then there exist $p, q \in \partial K$ such that $b-a = q-p$.*

Proof. If $a = b$ or $a, b \in \partial K$ the result follows trivially. Henceforth we assume $a \neq b$ and that at least a or b is an interior point of K. We pick a point $c \in K$ out of the line ab. Let P be the plane containing a, b, c and $W = K \cap P$. The set W is a convex body in P and the boundary of W in P is contained in ∂K. We take orthogonal coordinates (x, y) in P so that $(b-a)$ points into the positive direction of the y-axis. Let I be the orthogonal projection in P of W onto the x-axis.

Given $x \in I$, define the set $W(x)$ as $\{y \in \mathbb{R} : (x, y) \in W\}$. A simple application of Kuratowski criterion, see Theorem 1.8.8 in [**78**], implies that $W(x_i)$ converges to $W(x)$ in Hausdorff distance when x_i converges to x. Hence the function $x \in I \mapsto |W(x)|$ is continuous and takes a value larger than $||b-a||$ at the projection of a, b over the x-axis. If $|W(x)| = ||b-a||$ for some $x \in I$, we take as p, q the extreme points of the interval $W(x)$ chosen so that $q-p = b-a$ to conclude the proof. Otherwise, we would have $|W(x_0)| > ||b-a||$ at an extreme point x_0 of I. We may choose two points $p, q \in W(x_0)$ such that $|[p, q]| = ||b-a||$ and $q-p = b-a$. Since $W(x_0)$ is contained in the boundary of W in P, it is contained in ∂K and so $p, q \in \partial K$. □

Now we refine the results in Lemma 3.30 when K is strictly convex and has boundary of class C^ℓ_+, $\ell \geqslant 2$. We say that a convex body K is of class C^ℓ_+, $\ell \geqslant 1$, when ∂K is of class C^ℓ and its normal map $N_K : \partial K \to \mathbb{S}^1$ is a diffeomorphism of class $C^{\ell-1}$.

Corollary 3.32. *Let $K \subset \mathbb{R}^2$ be a convex body containing 0 as interior point. Then*

(1) *If $K \subset \mathbb{R}^2$ is strictly convex, then K_0 is strictly convex.*

(2) *If K is of class C^ℓ_+, $\ell \geqslant 2$, then K_0 is of class C^ℓ_+.*

Proof. To prove that K_0 is strictly convex, we take two different points $x_1 - x_2, y_1 - y_2 \in \partial K_0$, with $x_i, y_i \in K$, $i = 1, 2$. Then the four points belong to the boundary of K. For any $\lambda \in (0,1)$, we write the convex combination $\lambda(x_1 - x_2) + (1-\lambda)(y_1 - y_2)$ as

$$x_\lambda - y_\lambda = (\lambda x_1 + (1-\lambda)y_1) - (\lambda x_2 + (1-\lambda)y_2).$$

Since $x_1 \neq y_1$ or $x_2 \neq y_2$, the strict convexity of K implies that x_λ or y_λ is an interior point of K. Then $x_\lambda - y_\lambda$ is an interior point of K_0. Since $\lambda \in (0,1)$ and the boundary points are arbitrary, the set K_0 is strictly convex.

To prove the boundary regularity of K_0 we follow Schneider's arguments [**78**, p. 115] and observe that the support function h_K of K is defined, when $u \neq 0$, by

$$h_K(u) = \langle u, N_K^{-1}(u)\rangle,$$

where $N_K : \partial K \to \mathbb{S}^1$ is the Gauss map, a diffeomorphism of class $C^{\ell-1}$ since K is of class C^ℓ_+. By Corollary 7.1.3 in [**78**]

$$\nabla h_K(u) = N_K^{-1}\bigg(\frac{u}{|u|}\bigg), \tag{3.25}$$

and so h_K is of class C^ℓ. This implies that the support function of K_0, $h_{K_0}(u) = h_K(u) + h_K(-u)$, is of class C^ℓ. Hence the polar body K_0^* of K_0 has boundary of class C^ℓ. The Gauss map $N_{K_0^*}$ of K_0^* can be described as

$$N_{K_0^*} : \rho(K_0^*, u)u \mapsto \frac{N_K^{-1}(u)}{|N_K^{-1}(u)|},$$

where $\rho(K_0^*, \cdot) = h_K^{-1}(\cdot)$ is the radial function of K_0^*, of class $C^{\ell-1}$. Hence $N_{K_0^*}$ is a diffeomorphism of class $C^{\ell-1}$ and so K_0^* is of class C^ℓ_+. Now the support function of K_0^* is of class C^ℓ_+ and we reason in the same way interchanging the roles of K_0^* and K_0 to get the result. □

Remark 3.33. If $K \subset \mathbb{R}^2$ is a centrally symmetric convex body, for any $p \in \partial K$, the line passing through p and $-p$ divides K into two regions of equal area. Hence the line through 0 and $-2p$ divides $-p+K$ into two regions of the same area. When p moves along ∂K, the point $-2p$ parametrizes $\partial(2K)$.

Let K be a convex set of class C^ℓ_+, $\ell \geqslant 2$, $C = \partial K$ and $\gamma : \mathbb{R} \to \mathbb{R}^2$ an L-periodic clockwise arc-length parameterization of C, with $L = \text{length}(C)$. The set $K_0 = \bigcup_{p\in C}(-p + K)$ has smooth boundary C_0. For any $v \in \mathbb{R}$, we denote by $\gamma_v(u) = \gamma(u+v) - \gamma(v)$. Let $\Gamma_v = (\gamma_v, t_v)$ be the horizontal lifting of γ_v with $t_v(0) = 0$. If we call $\Omega_v(u)$ the planar region delimited by the segment $[0, \gamma_v(u)]$ and the restriction of γ_v to $[0,u]$ then a standard application of the Divergence Theorem to the vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ implies

$$t_v(u) = \int_0^u \langle \gamma_v, J(\dot\gamma_v)\rangle(\xi)\, d\xi = 2\,|\Omega_v(u)|.$$

Our next goal is to prove that $\mathbb{S}_K$ is the union of two graphs defined in K_0 of class C^2 and coinciding on ∂K_0.

Theorem 3.34. *Let $K \subset \mathbb{R}^2$ be a convex body with C^ℓ_+ boundary, $\ell \geqslant 2$. Then*

(1) *$\mathbb{S}_K$ is of class C^ℓ outside the poles.*

(2) *There exist two functions* $g_1, g_2 : K_0 \to \mathbb{R}$ *of class* C^ℓ *on* $\operatorname{int}(K_0)$ *such that*
$$\mathbb{S}_K = \operatorname{graph}(g_1) \cup \operatorname{graph}(g_2),$$
with $g_1 > g_2$ *on* $\operatorname{int}(K_0)$ *and* $g_1 = g_2$ *on* C_0. *This imples that* $\mathbb{S}_K$ *is an embedded surface.*

Moreover, if K *is centrally symmetric then* $g_1 + g_2 = 2|K|$ *and hence* $\mathbb{S}_K$ *is symmetric with respect to the horizontal Euclidean plane* $t = |K|$.

Definition 3.35. The domain delimited by the embedded sphere $\mathbb{S}_K$ is a ball $\mathbb{B}_K$ that we call the *Pansu-Wulff shape* of $||\cdot||_K$.

Proof of Theorem 3.34. That $\mathbb{S}_K$ is C^ℓ outside the singular set follows from the parameterization (3.22) since the function $r(s)$ is of class C^ℓ. This proves 1.

We break the proof of 2 into several steps. Recall that $C = \partial K$ and $C_0 = \partial K_0$.

Step 1. Given $x \in K_0 \backslash \{0\}$, we claim that $x \in C - p$ for some $p \in C$ if and only if the segment $[p, p + x]$ is contained in K and $p, p + x \in C$. This means that the number of curves $C - p$, with $p \in C$, passing through $x \neq 0$ coincides with the number of segments parallel to x of length $|x|$ and boundary points in C. This step is trivial.

Step 2. Given $x \in K_0 \setminus \{0\}$, the number of segments $[p, p + x]$ contained in K with $p, p+x \in C$ is either 1 or 2. The first case corresponds to maximal length and happens if and only if x belongs to C_0.

To prove this we consider $v = x/|x|$ and a line L orthogonal to v. For any z in L we consider the intersection $I_z = L_z \cap K$, where L_z is the line passing through z with direction v. The set $J = \{z \in L : I_z \neq \emptyset\}$ is a non-trivial segment in L. The strict convexity of K implies that the map $F : J \to \mathbb{R}$ defined by $F(z) = |I_z|$ is strictly concave. Since F vanishes at the extreme points of J, it has just one maximum point $z_0 \in \operatorname{int}(J)$ and each value in the interval $(0, F(z_0))$ is taken by two different points in J. The observation that there is a bijective correspondence between the segments $[p, p + x]$ contained in K with $p, p + x \in C$ and the points $z \in L$ with $F(z) = |x|$ proves the first part of the claim.

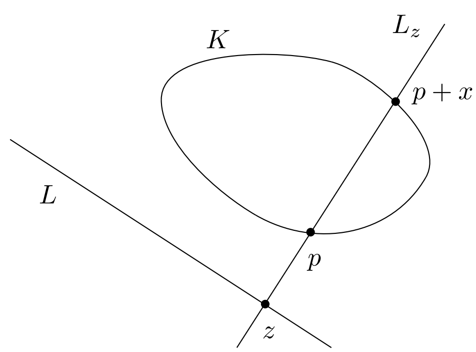


Figure 9. Construction of the map F

To prove the second part of the claim we fix some $x \in K_0$. We take $p \in C$ such that the segment $[p, p + x]$ is contained in K and $p, p + x \in C$. Assume first that $x \in C_0$. If there were a larger segment $[q, q + \mu x]$ contained in K with $q, q + \mu x \in C$

and $\mu > 1$ then we would have $\mu x \in C - q \subset K_0$, a contradiction. Hence the length of $[p, p+x]$ is is the largest possible in the direction of x. Assume now that the length of $[p, p+x]$ yields the maximum of length of intervals contained in K in the direction of x. If $x \notin C_0$ then x is an interior point of K_0 and, since $0 \in \operatorname{int}(K_0)$, there would exist $\lambda > 1$ such that $\lambda x \in K_0$. Hence there is some $q \in C$ such that $\lambda x \in C - p$ and the segment $[q, q + \lambda x] \subset C$ and has length larger than $|x|$, a contradiction that proves that $x \in C_0$.

Step 3. Given any point $x \in \operatorname{int}(K_0)$, there are exactly two points in $\mathbb{S}_K$ at heights $g_1(x) > g_2(x)$. In case K is centrally symmetric then $g_1(x) + g_2(x) = 2|K|$.

By the previous steps, there are exactly two points $p, q \in C$ so that $p+x, q+x \in C$ and the segments $[p, p+x], [q, q+x]$ are contained in K. We may assume that $p, p+x, q+x, q$ are ordered clockwise along C. The heights of the points in $\mathbb{S}_K$ projecting over x are given by twice the areas of the sets A and B, where A is determined by the portion of C from p to $p+x$ and the segment $[p+x, p]$, and B is determined by the portion of C from q to $q+x$ and the segment $[q+x, q]$. Since A is properly contained in B we have $g_2(x) = 2|A| < 2|B| = g_1(x)$.

In case K is centrally symmetric, the central symmetry maps $p+x$ to q and $q+x$ to p since $[p, p+x]$ and $[q, q+x]$ are the only segments in K of length $|x|$ with boundary points on C. Hence $|A| + |B| = |K|$ and so $g_1(x) + g_2(x) = 2|K|$.

Step 4. The functions g_1, g_2 are of class C^ℓ in $\operatorname{int}(K_0) \setminus \{0\}$.

This follows from the implicit function theorem since $\mathbb{S}_K$ is C^ℓ outside the poles. □

THEOREM 3.36. *Let $K \subset \mathbb{R}^2$ be a convex body of class C^2_+. Then $\mathbb{S}_K$ is of class C^2 around the poles.*

PROOF. We consider a horizontal lifting $\Gamma = (x, y, t)$ of a clockwise arc-length parametrization γ of ∂K. Then a parameterization of $\mathbb{S}_K$ is given by $(\mathbf{x}, \mathbf{y}, \mathbf{t})(u, v) = \ell_{-\Gamma(v)}(\Gamma(u+v))$. This means

$$\begin{aligned} \mathbf{x}(u,v) &= x(u+v) - x(v), \\ \mathbf{y}(u,v) &= y(u+v) - y(v), \\ \mathbf{t}(u,v) &= t(u+v) - t(v) - x(u+v)y(v) + y(u+v)x(v). \end{aligned} \tag{3.26}$$

The tangent vectors $\partial/\partial u, \partial/\partial u$ are the image of $(1,0)$ and $(0,1)$ under the parameterization and are given by

$$\begin{aligned} \frac{\partial}{\partial u} &= \dot{x}(u+v)\, X + \dot{y}(u+v)\, Y. \\ \frac{\partial}{\partial v} &= \big(\dot{x}(u+v) - \dot{x}(v)\big)\, X + \big(\dot{y}(u+v) - \dot{y}(v)\big)\, Y + h(u,v)\, T, \end{aligned}$$

where

$$h(u,v) = 2\big(\dot{x}(v)(y(u+v) - y(v)) - \dot{y}(v)(x(u+v) - x(v))\big). \tag{3.27}$$

Geometrically, $h(u,v)$ is the scalar product of the position vector $(x(u+v) - x(v), y(u+v) - y(v))$ with $J((\dot{x}, \dot{y}))$, that is always negative for $u > 0$. A Riemannian unit normal vector N can be easily computed from the expressions of $\partial/\partial u$ and $\partial/\partial v$ and is given by

$$N = \frac{h\big(\dot{y}(u+v)X - \dot{x}(u+v)Y\big) + gT}{\big(h^2 + g^2\big)^{1/2}}, \tag{3.28}$$

where

$$g(u,v)=\dot{x}(v)\dot{y}(u+v)-\dot{y}(v)\dot{x}(u+v). \tag{3.29}$$

We have

$$|N_h|=\frac{|h|}{\big(h^2+g^2\big)^{1/2}},\qquad \langle N,T\rangle=\frac{g}{\big(h^2+g^2\big)^{1/2}}$$

Let us see that $\mathbb{S}_K$ is a C^2 surface near the south pole $(0,0,0)$. The arguments for the north pole of are similar. To see that $\mathbb{S}_K$ is C^1 near the south pole, it is enough to check that N extends continuously to $u=0$. Let us see that

$$\lim_{(u,v)\to(0,v_0)} N(u,v)=-T. \tag{3.30}$$

Since $g<0$, from the expression (3.28) it is enough to prove that

$$\lim_{(u,v)\to(0,v_0)} \frac{h}{g}(u,v)=0. \tag{3.31}$$

Since x and y are functions of class C^2, we use Taylor expansions around v to get

$$\begin{aligned} x(u+v)&=x(v)+\dot{x}(v)u+R(u,v)u, & y(u+v)&=y(v)+\dot{y}(v)u+R(u,v)u,\\ \dot{x}(u+v)&=\dot{x}(v)+\ddot{x}(v)u+R(u,v)u, & \dot{y}(u+v)&=\dot{y}(v)+\ddot{y}(v)u+R(u,v)u. \end{aligned}$$

In the above equations R denotes a continuous functions of (u,v) (depending on the equation) that converges to 0 when $u\to 0$ independently of v. This follows from the integral expression for the reminder in Taylor's expansion. Then we have

$$\begin{aligned} \lim_{(u,v)\to(0,v_0)}\frac{h}{g}(u,v)&=\lim_{(u,v)\to(0,v_0)}\frac{R(u,v)u}{-\kappa(v)u+R(u,v)u}\\ &=\lim_{(u,v)\to(0,v_0)}\frac{R(u,v)}{-\kappa(v)+R(u,v)}=0, \end{aligned}$$

where

$$\kappa(v)=\big(\dot{y}\ddot{x}-\dot{x}\ddot{y}\big)(v)$$

is the (positive) geodesic curvature of γ. This proves (3.31) and so $\mathbb{S}_K$ is of class C^1 around $(0,0,0)$.

To prove that $\mathbb{S}_K$ is of class C^2 around the origin it is enough to show that the Riemannian second fundamental form of $\mathbb{S}_K$ converges to 0 when $(u,v)\to(0,v_0)$. We first compute

$$\lim_{(u,v)\to(0,v_0)} D_{\partial/\partial u}N.$$

Since

$$\begin{aligned} D_{\partial/\partial u}N&=\frac{\partial}{\partial u}\left(\frac{h\dot{y}(u+v)}{\sqrt{h^2+g^2}}\right)X-\frac{\partial}{\partial u}\left(\frac{h\dot{x}(u+v)}{\sqrt{h^2+g^2}}\right)Y+\frac{g}{\sqrt{h^2+g^2}}\,J(\tfrac{\partial}{\partial u})\\ &\quad+\left(\frac{\partial}{\partial u}\left(\frac{g}{\sqrt{h^2+g^2}}\right)+\frac{h}{\sqrt{h^2+g^2}}\right)T. \end{aligned} \tag{3.32}$$

A direct computation taking into account $\frac{\partial h}{\partial u}=2g$ yields

$$\frac{\partial}{\partial u}\left(\frac{h}{\sqrt{h^2+g^2}}\right)=\frac{2g^3-gh\frac{\partial g}{\partial u}}{(h^2+g^2)^{3/2}},\qquad \frac{\partial}{\partial u}\left(\frac{g}{\sqrt{h^2+g^2}}\right)=\frac{h^2\frac{\partial g}{\partial u}-2g^2h}{(h^2+g^2)^{3/2}}.$$

It is straightforward to check from the Taylor expressions that

$$\lim_{(u,v)\to(0,v_0)}\frac{h}{g^2}(u,v)=\lim_{(u,v)\to(0,v_0)}\frac{-\kappa(v_0)u^2+R(u,v)u^2}{\kappa(v_0)^2u^2+R(u,v)u^2}=\frac{-1}{\kappa(v_0)}.$$

Then we immediately get, dividing by $-g^3$,

$$\lim_{(u,v)\to(0,v_0)}\frac{\partial}{\partial u}\bigg(\frac{h}{\sqrt{h^2+g^2}}\bigg)=\lim_{(u,v)\to(0,v_0)}\frac{-2+\frac{h}{g^2}\frac{\partial g}{\partial u}}{((\frac{h}{g})^2+1)^{3/2}}=-1$$

and

$$\lim_{(u,v)\to(0,v_0)}\frac{\partial}{\partial u}\bigg(\frac{g}{\sqrt{h^2+g^2}}\bigg)=\lim_{(u,v)\to(0,v_0)}\frac{-\frac{h}{g}\frac{h}{g^2}\frac{\partial g}{\partial u}+2\frac{h}{g}}{((\frac{h}{g})^2+1)^{3/2}}=0.$$

Taking limits in (3.32) we get

$$\lim_{(u,v)\to(0,v_0)}D_{\partial/\partial u}N=J(\tfrac{\partial}{\partial u})-J(\tfrac{\partial}{\partial u})+0=0.$$

We complete $\frac{\partial}{\partial v}$ to an orthonormal basis of the tangent plane by adding the vector

$$E=\frac{\frac{\partial}{\partial v}-\langle\frac{\partial}{\partial u},\frac{\partial}{\partial v}\rangle\frac{\partial}{\partial u}}{(1-\langle\frac{\partial}{\partial u},\frac{\partial}{\partial v}\rangle^2)^{1/2}}.$$

Since $\lim_{(u,v)\to(0,v_0)}\frac{\partial}{\partial v}=0$, we have

$$\begin{aligned}\lim_{(u,v)\to(0,v_0)}D_EN&=\lim_{(u,v)\to(0,v_0)}D_{\partial/\partial v}N\\&=\lim_{(u,v)\to(0,v_0)}\bigg(-\frac{\partial}{\partial v}\bigg(\frac{h}{\sqrt{h^2+g^2}}\bigg)J(\tfrac{\partial}{\partial u})+\frac{\partial}{\partial v}\bigg(\frac{g}{(h^2+g^2)^{1/2}}\bigg)\bigg).\end{aligned}$$

A computation shows that

$$\frac{\partial}{\partial v}\bigg(\frac{h}{\sqrt{h^2+g^2}}\bigg)=\frac{g^2\frac{\partial h}{\partial v}-gh\frac{\partial g}{\partial v}}{(h^2+g^2)^{3/2}},\qquad\frac{\partial}{\partial v}\bigg(\frac{g}{\sqrt{h^2+g^2}}\bigg)=\frac{h^2\frac{\partial g}{\partial v}-gh\frac{\partial h}{\partial v}}{(h^2+g^2)^{3/2}}.$$

We trivially have

$$\lim_{(u,v)\to(0,v_0)}\frac{\partial h}{\partial v}(u,v)=\lim_{(u,v)\to(0,v_0)}\frac{\partial g}{\partial v}(u,v)=0.$$

Hence

$$\lim_{(u,v)\to(0,v_0)}\frac{\partial}{\partial v}\bigg(\frac{g}{\sqrt{h^2+g^2}}\bigg)=\lim_{(u,v)\to(0,v_0)}\frac{-\frac{h}{g}\frac{h}{g^2}\frac{\partial g}{\partial v}+\frac{h}{g^2}\frac{\partial h}{\partial v}}{((\frac{h}{g})^2+1)^{3/2}}=0.$$

On the other hand

$$\lim_{(u,v)\to(0,v_0)}\frac{\partial}{\partial v}\bigg(\frac{h}{\sqrt{h^2+g^2}}\bigg)=\lim_{(u,v)\to(0,v_0)}\frac{-\frac{1}{g}\frac{\partial h}{\partial v}+\frac{h}{g^2}\frac{\partial g}{\partial v}}{(h^2+g^2)^{3/2}}=0.$$

This equality holds from the Taylor expansions since

$$\lim_{(u,v)\to(0,v_0)}\frac{1}{g}\frac{\partial h}{\partial v}(u,v)=\lim_{(u,v)\to(0,v_0)}\frac{R(u,v)u}{-\kappa(v)u+R(u,v)u}=0.$$

So we conclude that $\lim_{(u,v)\to(0,v_0)}D_EN=0$. □

3.7. Minimization property of the Pansu-Wulff shapes. We prove in this section a minimization property satisfied by the balls $\mathbb{B}_K$. Let K be a convex body containing 0 in its interior. We assume that K is of class C^ℓ_+, with $\ell \geqslant 2$.

REMARK 3.37. Existence of isoperimetric regions in Carnot and nilpotent groups endowed with a sub-Finsler norm is proved in [**71**]. In the Heisenberg group $\mathbb{H}^1$ with a sub-Finsler norm this is done in [**34**, Thm. 3.1]. Proofs are based on Leonardi-Rigot's paper [**56**].

DEFINITION 3.38. Given $\mathbb{S}_K$, we let $g : K_0 \to \mathbb{R}$ be the function $g(x) = (g_1(x) + g_2(x))/2$, where g_1 and g_2 are the functions obtained in Theorem 3.34.

We also introduce the notation $\mathbb{S}_K^+ := \mathbb{S}_K \cap \{(x,t) : t \geqslant g(x)\}$, $\mathbb{S}_K^- := \mathbb{S} \cap \{(x,t) : t \leqslant g(x)\}$ and $D_0 = \{(x, g(x)) : x \in K_0\}$.

THEOREM 3.39. *Let $||\cdot||_K$ be the norm associated to a convex body $K \subset \mathbb{R}^2$ of class C^ℓ_+, with $\ell \geqslant 2$. Let $r > 0$ and $h : rK_0 \to \mathbb{R}$ a C^0 function. Consider a subset $E \subset \mathbb{H}^1$ with finite volume and finite K-perimeter such that*

$$\operatorname{graph}(h) \subseteq E \subset rK_0 \times \mathbb{R}.$$

Then

$$|\partial E|_K \geqslant |\partial \mathbb{B}_E|_K, \tag{3.33}$$

where $\mathbb{B}_E$ is the Wulff shape in $(\mathbb{H}^1, ||\cdot||_K)$ with $|E| = |\mathbb{B}_E|$.

PROOF. Let $g_r : rK_0 \to \mathbb{R}$ the function defined by $g_r(x) = r^2 g(\frac{1}{r}x)$, where g is the function in Definition 3.38. Let D be the graph of g_r. We know that D divides the Wulff shape $r\mathbb{S}_K$ into two parts $r\mathbb{S}_K^+$ and $r\mathbb{S}_K^-$. Let W^+ and W^- the vector fields in $rK_0 \times \mathbb{R} \setminus L$ defined by translating vertically the vector fields

$$\pi_K(\nu_0)\big|_{r\mathbb{S}_K^+}, \quad \pi_K(\nu_0)\big|_{r\mathbb{S}_K^-},$$

respectively. Here ν_0 is the horizontal unit normal to $\mathbb{S}_K$.

As a first step in the proof we are going to show that if $F \subset rK_0 \times \mathbb{R}$ is a set of finite volume and K-perimeter so that rel int$(D) \subset$ int(F), then the inequality

$$\tfrac{1}{r}|F| \leqslant \int_D \langle W^+ - W^-, N_D\rangle dD + |\partial F|_K \tag{3.34}$$

holds, where N_D is the Riemannian normal pointing down and dD is the Riemannian measure of D. Equality holds in (3.34) if and only if $W^+ = \pi_K(\nu_h)$ $|\partial_K F|$-a.e. on $F^+ = F \cap \{t \geqslant g_r\}$ and $W^- = \pi_K(\nu_h)$ $|\partial_K F|$-a.e. on $F^- = F \cap \{t \leqslant g_r\}$. Here ν_h is the horizontal unit normal to F.

To prove (3.34) we consider two families of functions. For $0 < \varepsilon < 1$ we consider smooth functions φ_ε, depending on the Riemannian distance to the vertical axis $L = \{x = y = 0\}$, so that $0 \leqslant \varphi_\varepsilon \leqslant 1$ and

$$\begin{aligned} \varphi_\varepsilon(p) &= 0, & d(p,L) &\leqslant \varepsilon^2, \\ \varphi_\varepsilon(p) &= 1, & d(p,L) &\geqslant \varepsilon, \\ |\nabla \varphi_\varepsilon(p)| &\leqslant 2/\varepsilon, & \varepsilon^2 \leqslant d(p,L) &\leqslant \varepsilon. \end{aligned}$$

Again for $0<\varepsilon<1$ we consider smooth functions ψ_ε, depending on the Riemannian distance to the Euclidean hyperplane $\Pi_0=\{t=0\}$, so that $0\leqslant\psi_\varepsilon\leqslant 1$ and

$$\begin{aligned}\psi_\varepsilon(p)&=1, &\quad d(p,\Pi_0)&\leqslant\varepsilon^{-1/2},\\ \psi_\varepsilon(p)&=0, &\quad d(p,\Pi_0)&\geqslant\varepsilon^{-1/2}+1,\\ |\nabla\psi_\varepsilon(p)|&\leqslant 2, &\quad \varepsilon^{-1/2}\leqslant d(p,\Pi_0)&\leqslant\varepsilon^{-1/2}+1.\end{aligned}$$

For any $\varepsilon>0$, the vector field $\varphi_\epsilon\psi_\varepsilon W$ has compact support.

It is easy to prove that F^+ and F^- have finite K-perimeter. Since F^+ has also finite (sub-Riemannian) perimeter, applying the Divergence Theorem to F^+ and the horizontal vector field $\varphi_\epsilon\psi_\varepsilon W^+$, we have

$$\begin{aligned}\int_{F^+}\operatorname{div}(\varphi_\epsilon\psi_\varepsilon W^+)d\mathbb{H}^1&=\int_D\langle\varphi_\epsilon\psi_\varepsilon W^+,N_D\rangle dD\\&\quad+\int_{\{t>g_r\}}\langle\varphi_\epsilon\psi_\varepsilon W^+,\nu_h\rangle d|\partial F|.\end{aligned}\tag{3.35}$$

Where N_D is the Riemannian unit normal to D pointing into F^-, dD is the Riemannian area element on D, and ν_h is the outer horizontal unit normal to F.

We take limits in the left hand side of Equation (3.35) when $\varepsilon\to 0$. We write

$$\int_{F^+}\operatorname{div}(\varphi_\epsilon\psi_\varepsilon W^+)d\mathbb{H}^1=\int_{F^+}\varphi_\epsilon\psi_\varepsilon\operatorname{div}W^+d\mathbb{H}^1+\int_{F^+}\langle\nabla(\varphi_\epsilon\psi_\varepsilon),W^+\rangle d\mathbb{H}^1.\tag{3.36}$$

Since $\langle\varphi_\varepsilon\nabla\psi_\varepsilon,W^+\rangle$ is bounded and converges pointwise to 0, and

$$\int_{F^+}\langle\psi_\varepsilon\nabla\varphi_\varepsilon,W^+\rangle\leqslant\int_{\{(x,t):\varepsilon^2<|x|<\varepsilon,\ 0<t<\varepsilon^{-1/2}+1\}}\psi_\varepsilon|\nabla\varphi_\epsilon|d\mathbb{H}^1,$$

we have

$$\lim_{\varepsilon\to 0}\int_{F^+}\langle\nabla(\varphi_\epsilon\psi_\varepsilon),W^+\rangle d\mathbb{H}^1=0.\tag{3.37}$$

On the other hand, $\operatorname{div}W^+=\frac{1}{r}$, the mean curvature of $r\mathbb{B}_K$. We consider the orthonormal vectors $Z=-J(\nu_h)$, $E=\langle N,T\rangle\,\nu_h-|\nu_h|\,T$ and N, globally defined on $(rK_0\times\mathbb{R})\setminus L$ by vertical translations. We know from Lemma 3.5 that

$$\langle D_ZW^+,Z\rangle=\tfrac{1}{r},\qquad\langle D_EW^+,E\rangle=2\,\langle N,T\rangle|N_h|\,\langle W^+,J(\nu_h)\rangle.$$

It remains to compute $\langle D_NW^+,N\rangle$. We express $N=\lambda E+\mu T$ as a linear combination of E and T, where $\lambda=|N_h|/\langle N,T\rangle$, $\mu=1/\langle N,T\rangle$. Observe that $\langle N,T\rangle\neq 0$ on $\operatorname{int}(K_0)$ since $r\mathbb{S}_K^+$ is a t-graph. So we have

$$\begin{aligned}\langle D_NW^+,N\rangle&=\lambda\langle D_EW^+,N\rangle+\mu\langle D_TW^+,N\rangle\\&=\lambda^2\langle D_EW^+,E\rangle+\lambda\mu\langle D_EW^+,T\rangle+\mu\langle J(W^+),N_h\rangle\\&=\lambda^2\langle D_EW^+,E\rangle-\lambda\mu\langle N,T\rangle\langle W^+,J(\nu_h)\rangle-\mu|N_h|\langle W^+,J(\nu_h)\rangle\\&=\left(\frac{|N_h|}{\langle N,T\rangle}\right)^2\langle D_EW^+,E\rangle-\frac{1}{\langle N,T\rangle^2}\,\langle D_EW^+,E\rangle\\&=\langle D_EW^+,E\rangle,\end{aligned}$$

where we have used that $D_TW^+=J(W^+)$ since W^+ is a linear combination of W^+,Y multiplied by functions that do not depend on t. Hence

$$\operatorname{div}W^+=\langle D_ZW^+,Z\rangle+\langle D_EW^+,E\rangle+\langle D_NW^+,N\rangle=\frac{1}{r}$$

on $\operatorname{int}(K_0)$. Since $\varphi_\epsilon\psi_\varepsilon \operatorname{div} W^+$ is uniformly bounded, F^+ has finite volume and $\lim_{\varepsilon\to 0}\varphi_\epsilon\psi_\varepsilon = 1$, we can apply Lebesgue's Dominated Convergence Theorem to get

$$\lim_{\varepsilon\to 0}\int_{F^+}\varphi_\epsilon\psi_\varepsilon \operatorname{div} W^+ d\mathbb{H}^1 = \tfrac{1}{r}|F^+|. \tag{3.38}$$

So we get from (3.36), (3.37) and (3.38)

$$\lim_{\varepsilon\to 0}\int_{F^+}\operatorname{div}(\varphi_\varepsilon\psi_\varepsilon W^+)\, d\mathbb{H}^1 = \tfrac{1}{r}\,|F^+|. \tag{3.39}$$

Now we treat the remainings terms in (3.35). Using the representation of perimeter obtained in (2.9) for sets of finite K-perimeter sets we have

$$\int_{\{t>g_r\}}\langle W^+,\nu_h\rangle d|\partial F| \leqslant \int_{\{t>g_r\}}\|\nu_h\|_* d|\partial F| = |\partial F^+|_K, \tag{3.40}$$

with equality if and only if $W^+=\pi(\nu_h)$ $|\partial F|$-a.e. on $\{t>g_r\}$. From equations (3.39) and (3.40), taking limits in Equation (3.35) when $\varepsilon\to 0$,

$$\tfrac{1}{r}|F^+| \leqslant \int_D \langle W^+, N_D\rangle dD + |\partial F^+|_K, \tag{3.41}$$

with equality if and only if $W^+=\pi(\nu_h)$ $|\partial F|$-a.e. on $\partial F\cap\{t>g_r\}$.

We consider now the foliation of $rK_0\times\mathbb{R}$ by vertical translations of $r\mathbb{S}_K^-$. Reasoning as in the previous case we get

$$\tfrac{1}{r}|F^-| \leqslant -\int_D \langle W^-, N_D\rangle dD + |\partial F^-|_K. \tag{3.42}$$

with equality if and only if $W^-=\pi(\nu_h)$ $|\partial F|$-a.e. on $\partial\{t<g_r\}$. Hence, adding (3.41) and (3.42), and taking into account $|\partial F|_K(\mathbb{H}^1\setminus D) = |\partial F|_K$ and that $F\cap D$ does not contribute to the volume of F, we get

$$\tfrac{1}{r}|F| \leqslant \int_D \langle W^+-W^-, N_D\rangle dD + |\partial F|_K,$$

and so (3.34) holds, with equality if and only if equalities (3.41) and (3.42) hold. This completes the first part of the proof.

Recall that $h: rK_0\to\mathbb{R}$ is a function so that $D=\operatorname{graph}(h)\subset E$. We take two values $t_m<t_M$ such that

$$h+t_m<g_r<h+t_M.$$

We apply inequality (3.34) to the set $B=B^-\cup B^0\cup B^+$, where

- $B^0=\{(x,t): x\in rK_0, |t-g_r|\leqslant (t_M-t_m)/2\}$,
- $B^+=r\mathbb{B}_K^+ + (0,(t_M-t_m)/2)$,
- $B^-=r\mathbb{B}_K^- - (0,(t_M-t_m)/2)$.

By construction, $D=\operatorname{graph}(g_r)\subset B^0$. Since the lateral boundary of B^0 is contained in $\partial(rK_0\times\mathbb{R})$ and the outer unit normal to $\partial(rK_0\times\mathbb{R})$ coincides with W^+ and W^-, the lateral K-boundary area of B^0 is equal to

$$(t_M-t_m)\int_{\partial(rK_0)}||\nu_0||_* d(\partial(rK_0)),$$

where $d(\partial(rK_0))$ is the Riemannian length element of the C^1 curve $\partial(rK_0)$. Hence we get

$$|\partial B|_K = (t_M-t_m)\int_{\partial(rK_0)}\|\nu_0\|_* d(\partial(rK_0)) + |\partial(r\mathbb{B}_K)|_K.$$

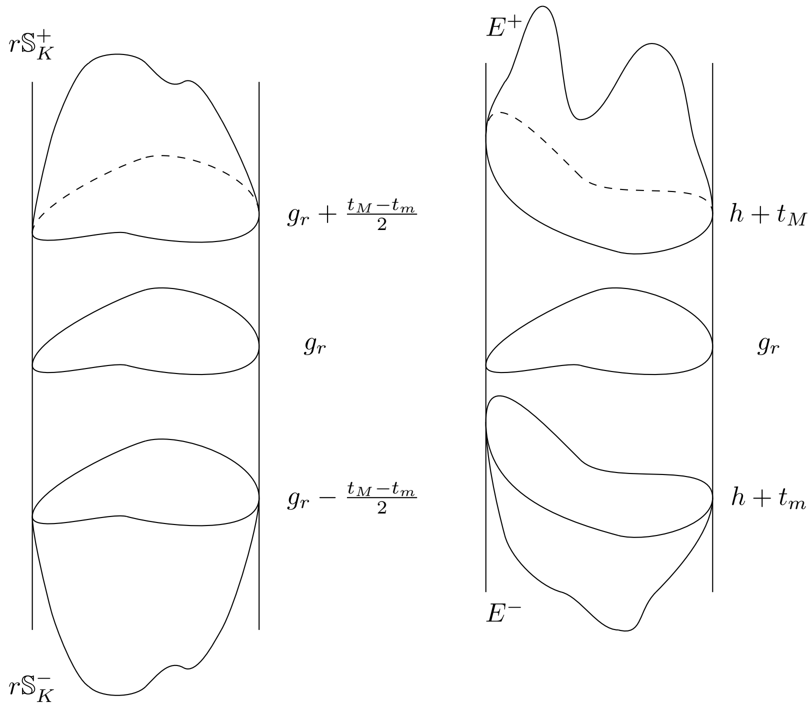


FIGURE 10. Geometric construction in the proof of Theorem 3.39

On the other hand, since

$$|B| = |r\mathbb{B}_K| + |rK_0|(t_M - t_m),$$

we obtain

$$\begin{aligned}\tfrac{1}{r}(|r\mathbb{B}_K| + |rK_0|(t_M - t_m)) &= \int_D \langle W^+ - W^-, N_D\rangle dD \\ &\quad + (t_M - t_m)\int_{\partial(rK_0)} \|\nu_0\|_* d\partial(rK_0) + |\partial(r\mathbb{B}_K)|_K.\end{aligned} \tag{3.43}$$

Now we apply (3.34) to the set $\mathbb{E}$ consisting on the union of $E^+ = E \cap \{t \geqslant h\}$ translated by the vector $(0, t_M)$, $E^- = E \cap \{t \leqslant h\}$ translated by the vector $(0, t_m)$ and the vertical filling in between the two sets. We reason as before to get

$$\begin{aligned}\tfrac{1}{r}(|E| + |rK_0|(t_M - t_m)) &\leqslant \int_D \langle W^+ - W^-, N_D\rangle dD \\ &\quad + (t_M - t_m)\int_{\partial D} \|\nu_0\|_* d\partial D_0 + |\partial E|_K.\end{aligned} \tag{3.44}$$

From (3.43) and (3.44) we get

$$|\partial E|_K \geqslant |\partial(r\mathbb{B}_K)|_K + \tfrac{1}{r}(|E| - |r\mathbb{B}|).$$

Let $f(\rho) = |\partial(\rho\mathbb{B}_K)|_K + \frac{1}{\rho}(|E| - |\rho\mathbb{B}|)$. Since $\rho\mathbb{B}_K$ has mean curvature $\frac{1}{\rho}$, Theorem 3.16 guarantees that the Wulff shape $\rho\mathbb{B}_K$ is a critical point of $A - \frac{1}{\rho}|\cdot|$

for any variation. Therefore $|\partial(\rho\mathbb{B}_K)|'_K - \frac{1}{\rho}|\rho\mathbb{B}_K|' = 0$ where primes indicates the derivative with respect to ρ. Hence we have

$$f'(\rho) = -\tfrac{1}{\rho^2}(|E| - |\rho\mathbb{B}_K|).$$

So the only critical point of f corresponds to the value ρ_0 so that $|\rho_0\mathbb{B}_K| = |E|$. Since the function $\rho \mapsto |\rho B_K|$ is strictly increasing and takes its values in $(0, +\infty)$, we obtain that $f(\rho)$ is a convex function with a unique minimum at ρ_0. Hence we obtain

$$|\partial E|_K \geqslant f(r) \geqslant f(\rho_0) = |\partial(r_0\mathbb{B}_K)|_K,$$

which implies (3.33). □

4. Regularity of sets with prescribed mean curvature

4.1. Regularity of sets with prescribed mean curvature.

4.1.1. *Sets with prescribed mean curvature.* Consider an open set $\Omega \subset M$, and an integrable function $f \in L^1_{loc}(\Omega)$. We say that a set of locally finite K-perimeter $E \subset \Omega$ has *prescribed K-mean curvature f in* Ω if, for any bounded open set $B \subset \Omega$, E is a critical point of the functional

$$P_K(E, B) - \int_{E\cap B} f\, d\mathbb{H}^1. \tag{4.1}$$

If $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface then S has prescribed K-mean curvature f if it is a critical point of the functional

$$A_K(S \cap B) - \int_{E\cap B} f\, d\mathbb{H}^1, \tag{4.2}$$

for any bounded open set $B \subset \Omega$.

If E has boundary $S = \partial E \cap \Omega$ of class C^2, standard arguments imply that E has prescribed K-mean curvature f in Ω if and only if $H_K = f$, where H_K is the K-mean curvature

$$H_K = \langle D_Z \pi_K(\nu_h), Z\rangle,$$

and ν_h is the *outer* horizontal unit normal, see [**72**]. Since by [**72**, Lemma 2.1] the Levi-Civita connection D and the pseudo-hermitian connection ∇ coincide for horizontal vector fields, we obtain that

$$H_K = \langle D_Z \pi_K(\nu_h), Z\rangle = \langle \nabla_Z \pi_K(\nu_h), Z\rangle.$$

It is important to remark that the mean curvature H_K strongly depends on the choice of ν_h. When K is centrally symmetric, $\pi_K(-u) = -\pi_K(u)$ and so the mean curvature changes its sign when we take $-\nu_h$ instead of ν_h. When K is not centrally symmetric, there is no relation between the mean curvatures associated to ν_h and $-\nu_h$.

A set $E \subset \mathbb{H}^1$ with Euclidean Lischiptz boundary has locally finite K-perimeter: we know that it has locally bounded sub-Riemannian perimeter by Proposition 2.14 in [**36**] and we can apply the perimeter estimates in § 2.3. Letting $\mathcal{H}^2$ be the Riemannian 2-dimensional Hausdorff measure, the Riemannian outer unit normal N is defined $\mathcal{H}^2$-a.e. in ∂E, and it can be proven that

$$P_K(E, V) = \int_{\partial E\cap V} ||N_h||_{K,*}\, d\mathcal{H}^2. \tag{4.3}$$

We say that a set E of locally finite K-perimeter in an open set $\Omega \subset \mathbb{H}^1$ has *constant* prescribed K-mean curvature if there exists $\lambda \in \mathbb{R}$ such that E has prescribed K-mean curvature λ. This means that E is a critical point of the functional $E \mapsto P_K(E, B) - \lambda|E \cap B|$ for any bounded open set $B \subset \Omega$.

Our next result implies that Euclidean Lipschitz isoperimetric boundaries (for the K-perimeter) have constant prescribed K-mean curvature.

PROPOSITION 4.1. *Let $E \subset \mathbb{H}^1$ be a bounded set with Euclidean Lipschitz boundary. Assume that E a critical point of the K-perimeter for variations preserving the volume of E up to first order. Let $\Omega \subset \mathbb{H}^1$ be an open set so that $\Omega \cap S_0 = \emptyset$ and $P_K(E, \Omega) > 0$. Then E has constant prescribed K-mean curvature in Ω.*

PROOF. Since the $K-$perimeter of E in Ω is positive there exists a horizontal vector field U_0 with compact support in Ω so that $\int_E \operatorname{div} U_0 \, d\mathbb{H}^1 > 0$. Let $\{\psi_s\}_{s\in\mathbb{R}}$ be the flow associated to U_0 and define

$$H_0 = \frac{\frac{d}{ds}|_{s=0} \, A_K(\psi_s(S))}{\frac{d}{ds}|_{s=0}|\psi_s(E)|}. \tag{4.4}$$

Let W any vector field with compact support in Ω and associated flow $\{\varphi_s\}_{s\in\mathbb{R}}$. Choose $\lambda \in \mathbb{R}$ so that $W - \lambda U_0$ satisfies

$$\frac{d}{ds}\bigg|_{s=0} |\varphi_s(E)| - \lambda \, \frac{d}{ds}\bigg|_{s=0} |\psi_s(E)| = 0.$$

This means that the flow of $W - \lambda U_0$ preserves the volume of E up to first order.

By our assumption on E we get

$$Q(W - \lambda U_0) = 0,$$

where Q is defined in (4.5). Now Lemma 4.2 implies $Q(W) = \lambda Q(U_0)$ and, from the definition of H_0, we get

$$Q(W) = \lambda Q(U_0) = \lambda H_0 \frac{d}{ds}\bigg|_{s=0} |\psi_s(E)| = H_0 \frac{d}{ds}\bigg|_{s=0} |\varphi_s(E)|.$$

This implies that E is a critical point of the functional $E \mapsto |\partial E|_K - H_0|E|$ and so it has prescribed K-mean curvature equal to the constant H_0. □

LEMMA 4.2. *Let $E \subset \mathbb{H}^1$ be a bounded set with Euclidean Lipschitz boundary S. Let $\Omega \subset \mathbb{H}^1$ be an open set such that $\Omega \cap S_0 = \emptyset$. Let U be a vector field with compact support Ω and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated flow. Then the derivative*

$$Q(U) = \frac{d}{ds}\bigg|_{s=0} A_K(\varphi_s(S)) \tag{4.5}$$

exists and is a linear function of U.

PROOF. For every $s \in \mathbb{R}$, the set $\varphi_s(E)$ has Euclidean Lipschitz boundary and so it has finite K-perimeter. By Rademacher's Theorem, the set

$$B = \{p \in S : S \text{ is not differentiable at } p\}$$

has $\mathcal{H}^2$-measure equal to 0.

For any $p \in S \setminus B$ we take the curve $\sigma(s) = \varphi_s(p)$. For every $s \in \mathbb{R}$ the surface $\varphi_s(S)$ is differentiable at $\sigma(p)$ and the vector field $W(s) = ((N_s)_h)_{\sigma(s)}$, where N_s

is the outer unit normal to $\varphi_s(\partial E)$, is differentiable along the curve σ. Let us estimate the quotient

$$\frac{||W(s+h)||_{K,*} - ||W(s)||_{K_*}}{h}. \tag{4.6}$$

Writing $W(s) = f(s)X_{\sigma(s)} + g(s)Y_{\sigma(s)}$ we have $||W(s)||_{K,*} = ||(f(s), g(s))||$, where $||\cdot||$ is the planar asymmetric norm associated to the convex set K. We have

$$\begin{aligned}\big|||W(s+h)||_{K,*} - ||W(s)||_{K_*}\big| &\leqslant ||(f(s+h)-f(s), g(s+h)-g(s))|| \\ &\leqslant C\left(|f(s+h)-f(s)| + |g(s+h)-g(s)|\right),\end{aligned}$$

for a constant $C > 0$ that only depends on K. The derivates of f and g can be estimated in terms of the covariant derivative $\frac{D}{ds}W = \frac{D}{ds}(N_s)_h$ along σ. Since

$$\left|\frac{D}{ds}(N_s)_h\right| \leqslant \left|\operatorname{div}_{\varphi_s(S)}(U)\right|$$

we get an uniform estimate on the derivatives of f and g independent of p. So the quotient (4.6) is uniformly bounded above by a constant independent of p.

To compute the derivative of $A_K(\varphi_s(S))$ at $s = 0$ we write

$$A_K(\varphi_s(S)) = \int_S \left(||(N_s)_h||_{K,*} \circ \varphi_s\right) \operatorname{Jac}(\varphi_s)\, d\mathcal{H}^2$$

The uniform estimate of the quotient (4.6) allows us to apply Lebesgue's dominated convergence theorem and Leibniz's rule to compute the derivative of $A_K(\varphi_s(S))$, given by

$$\int_S \frac{d}{ds}\bigg|_{s=0} \Big(\left(||(N_s)_h|| \circ \varphi_s\right) \operatorname{Jac}(\varphi_s)\Big)\, d\mathcal{H}^2.$$

Given a point $p \in (S \smallsetminus B) \cap \operatorname{supp}(U)$, since $\operatorname{supp}(U) \subset \Omega$ and $\Omega \cap S_0 = \emptyset$ we get $(N_h)_p \neq 0$ and so

$$\begin{aligned}\frac{D}{ds}\bigg|_{s=0} ||(N_s)_h||_{K,*}(\sigma(s)) &= \frac{D}{ds}\bigg|_{s=0} \langle (N_s)_h, \pi_K((N_s)_h)\rangle(\sigma(s)) \\ &= \langle \frac{D}{ds}\bigg|_{s=0}(N_s)_h, (N_h)_p\rangle + \langle (N_h)_p, (d\pi_K)\bigg(\frac{D}{ds}\bigg|_{s=0}(N_s)_h\bigg)\rangle.\end{aligned}$$

Since

$$\frac{D}{ds}\bigg|_{s=0}(N_s)_h = \frac{D}{ds}\bigg|_{s=0} N - \langle \frac{D}{ds}\bigg|_{s=0} N, T\rangle\, T,$$

and

$$\frac{D}{ds}\bigg|_{s=0} N = \sum_{i=1}^{2} \langle N_p, \nabla_{e_i} U\rangle\, e_i,$$

where e_i is an orthonormal basis of $T_p(\partial E)$, we get that

$$\frac{D}{ds}\bigg|_{s=0} ||N_s||_{K,*}$$

is a linear function $L(U)$ of U. □

REMARK 4.3. Proposition 4.1 can be applied to isoperimetric regions in $\mathbb{H}^1$ with Euclidean Lipschitz boundary. Of course, the regularity of isoperimetric regions in $\mathbb{H}^1$ is still an open problem.

4.1.2. *Intrinsic Euclidean Lipschitz graphs on a vertical plane in* $\mathbb{H}^1$. We denote by $\mathrm{Gr}(u)$ the *intrinsic* graph (Riemannian normal graph) of the Lipschitz function $u : D \to \mathbb{R}$, where D is a domain in a vertical plane. Using Euclidean rotations about the vertical axis $x = y = 0$, that are isometries of the Riemannian metric g, we may assume that D is contained in the plane $y = 0$. Since the vector field Y is a unit normal to this plane, the intrinsic graph $\mathrm{Gr}(u)$ is given by $\{\exp_p(u(p)Y_p) : p \in D\}$, where exp is the exponential map of g, and can be parameterized by the map

$$\Phi^u(x,t) = (x, u(x,t), t - xu(x,t)).$$

The tangent plane to any point in $S = \mathrm{Gr}(u)$ is generated by the vectors

$$\begin{aligned}\Phi^u_x &= (1, u_x, -u - xu_x) = X + u_x Y - 2uT,\\ \Phi^u_t &= (0, u_t, 1 - xu_t) = u_t Y + T\end{aligned}$$

and the characteristic direction is given by $Z = \tilde{Z}/|\tilde{Z}|$ where

$$\tilde{Z} = X + (u_x + 2uu_t)Y. \tag{4.7}$$

A unit normal to S is given by $N = \tilde{N}/|\tilde{N}|$ where

$$\tilde{N} = \Phi^u_x \times \Phi^u_t = (u_x + 2uu_t)X - Y + u_t T$$

and $\mathrm{Jac}(\Phi^u) = |\Phi^u_x \times \Phi^u_t| = |\tilde{N}|$. Therefore the horizontal projection of the unit normal to S is given by $N_h = \tilde{N}_h/|\tilde{N}|$, where $\tilde{N}_h = (u_x + 2uu_t)X - Y$. Observe that $J(Z) = -\nu_h$.

We also assume that $S = \mathrm{Gr}(u)$ is an $\mathbb{H}$-regular surface, meaning that $\tilde{N}_h$ and $\tilde{Z}$ in (4.7) and are continuous. Hence also $(u_x + 2uu_t)$ is continuous.

Remark 4.4. Let $\gamma(s) = (x,t)(s)$ be a C^1 curve in D then

$$\Gamma(s) = (x, u(x,t), t - xu(x,t))(s) \subset \mathrm{Gr}(u)$$

is also C^1 and

$$\Gamma'(s) = x'X + (x'u_x + t'u_t)Y + (t' - 2ux')T.$$

In particular horizontal curves in $\mathrm{Gr}(u)$ satisfy the ordinary differential equation

$$t' = 2u(x,t)x'. \tag{4.8}$$

From (2.15), the sub-Finsler K-area for a Euclidean Lipschitz surface S is

$$A_K(S) = \int_S \|N_h\|_{K,*} dS,$$

where $\|N_h\|_{K,*} = \langle N_h, \pi(N_h)\rangle$ with $\pi = (\pi_1, \pi_2) = \pi_K$ and dS is the Riemannian area measure. Therefore when we consider the intrinsic graph $S = \mathrm{Gr}(u)$ we obtain

$$\begin{aligned}A(\mathrm{Gr}(u)) &= \int_D \langle \tilde{N}_h, \pi(\tilde{N}_h)\rangle \; dxdt\\ &= \int_D (u_x + 2uu_t)\pi_1(u_x + 2uu_t, -1) - \pi_2(u_x + 2uu_t, -1)\, dxdt.\end{aligned}$$

Observe that the K-perimeter of a set was defined in terms of the *outer* unit normal. Hence we are assuming that S is the boundary of the *epigraph* of u.

Given $v \in C_0^\infty(D)$, a straightforward computation shows that

$$\frac{d}{ds}\Big|_{s=0} A(\mathrm{Gr}(u + sv)) = \int_D (v_x + 2vu_t + 2uv_t) M dxdt, \tag{4.9}$$

where

$$M = F(u_x + 2uu_t), \tag{4.10}$$

and F is the function

$$F(x) = \pi_1(x, -1) + x\frac{\partial \pi_1}{\partial x}(x, -1) - \frac{\partial \pi_2}{\partial x}(x, -1). \tag{4.11}$$

Since $(u_x + 2uu_t)$ is continuous and π is at least C^1 the function M is continuous.

4.2. Characteristic curves are C^2. Here we prove our main result, that characteristic curves in an intrinsic Euclidean Lipschitz $\mathbb{H}$-regular surface with continuous prescribed K-mean curvature are of class C^2. The reader is referred to Theorem 4.1 in [**42**] for a proof of the the sub-Riemannian case. The proof of Theorem 4.5 depends on Lemmas 4.6 and 4.7.

THEOREM 4.5. *Let K be a C^2_+ convex set in $\mathbb{R}^2$ with $0 \in \text{int}(K)$ and $||\cdot||_K$ the associated left-invariant norm in $\mathbb{H}^1$. Let $\Omega \subset \mathbb{H}^1$ be an open set and $E \subset \Omega$ a set of prescribed K- mean curvature $f \in C^0(\Omega)$ with an Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then the characteristic curves of $S \cap \Omega$ are of class C^2.*

PROOF. By the Implicit Function Theorem for $\mathbb{H}$-regular surfaces, see Theorem 6.5 in [**36**], given a point $p \in S$, after a rotation about the vertical axis, there exists an open neighborhood $B \subset \mathbb{H}^1$ of p such that $B \cap S$ is the intrinsic graph $\text{Gr}(u)$ of a function $u : D \to \mathbb{R}$, where D is a domain in the vertical plane $y = 0$, and $B \cap E$ is the epigraph of u. The function u is Euclidean Lipschitz by our assumption. Since $\text{Gr}(u)$ has prescribed continuous mean curvature f, from equation (4.9) we get

$$\int_D (v_x + 2vu_t + 2uv_t)M + fv\, dxdt = 0, \tag{4.12}$$

for each $v \in C_0^\infty(D)$. The function M is defined in (4.10). By Remark 4.3 in [**42**] implies that (4.12) holds for each $v \in C_0^0(D)$ for which $v_x + 2uv_t$ exists and is continuous.

Let $\Gamma(s)$ be a characteristic horizontal curve passing through p whose velocity is the vector field $\tilde{Z}$ defined in (4.7), that only depends on $u_x + 2uu_t$. Since S is $\mathbb{H}$-regular the function $u_x + 2uu_t$ is continuous and $\Gamma(s)$ is of class C^1. Let us consider the function F defined in (4.11) and define

$$g(s) = (u_x + 2uu_t)_{\Gamma(s)}.$$

Hence $F(g(s)) = M(s)$. The function F is C^1 for any convex set K of class C^2_+ and, from Lemma 4.6, we obtain that $F'(x) > 0$ for each $x \in \mathbb{R}$. Therefore F^{-1} is also C^1 and $g(s) = F^{-1}(M(s))$. Thanks to Lemma 4.7 we obtain that M is C^1 along Γ and we conclude that also g is C^1 along Γ. So $\tilde{Z}$ is C^1 and the curve Γ is C^2. □

LEMMA 4.6. *Let $K \subset \mathbb{R}^2$ be a convex body of class C^2_+ such that $0 \in \text{int}(K)$. Then the function F defined in (4.11) is C^1 and $F'(x) > 0$ for each $x \in \mathbb{R}$.*

PROOF. Parameterize the lower part of the boundary of the convex body K by a function ϕ defined on a closed interval $I \subset \mathbb{R}$. The function ϕ is of class C^2

in $\mathring{I}$ and the graph becomes vertical at the endpoints of I. As K is of class C^2_+ we have $\phi''(x) > 0$ for each $x \in \mathbb{R}$. Take $x \in \mathbb{R}$, then we have

$$\pi(x,-1) = N_K^{-1}\left(\frac{(x,-1)}{\sqrt{1+x^2}}\right),$$

where N_K is the outer unit normal to ∂K. Let $\varphi(x) \in \mathring{I}$ be the point where

$$(\varphi(x), \phi(\varphi(x))) = \pi(x,-1).$$

Therefore, if we consider the normal N_K of the previous equality we obtain

$$\frac{(\phi'(\varphi(x)),-1)}{\sqrt{1+(\phi'(\varphi(x)))^2}} = \frac{(x,-1)}{\sqrt{1+x^2}}.$$

Hence $\phi'(\varphi(x)) = x$ and so φ is the inverse of ϕ', that is invertible since $\phi''(x) > 0$ for each $x \in \mathbb{R}$. Notice that

$$\begin{aligned} F(x) &= \pi_1(x,-1) + x\frac{\partial \pi_1}{\partial x}(x,-1) - \frac{\partial \pi_2}{\partial x}(x,-1) \\ &= \varphi(x) + x\varphi'(x) - \phi'(\varphi(x))\varphi'(x) = \varphi(x), \end{aligned}$$

since $\phi'(\varphi(x)) = x$. Hence we obtain

$$F'(x) = \varphi'(x) = \frac{1}{\phi''(\varphi(x))} > 0$$

for each $x \in \mathbb{R}$. □

Lemma 4.7. *Let $\Omega \subset \mathbb{H}^1$ be an open set and $E \subset \Omega$ a set of prescribed K-mean curvature $f \in C^0(\Omega)$ with Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then the function M defined in (4.10) is of class C^1 along characteristic curves. Moreover, the differential equation*

$$\frac{d}{ds} M(\gamma(s)) = f(\gamma(s))$$

is satisfied along any characteristic curve γ.

Proof. Let $\Gamma(s)$ be a characteristic curve passing through p in $\mathrm{Gr}(u)$. Let $\gamma(s)$ be the projection of $\Gamma(s)$ onto the xt-plane, and $(a,b) \in D$ the projection of p to the xt-plane. We parameterize γ by $s \to (s, t(s))$. By Remark 4.4 the curve $s \to (s,t(s))$ satisfies the ordinary differential equation $t' = 2u$. For ε small enough, Picard-Lindelöf's theorem implies the existence of $r > 0$ and a solution $t_\varepsilon :]a-r, a+r[\to \mathbb{R}$ of the Cauchy problem

$$\begin{cases} t_\varepsilon'(s) = 2u(s, t_\varepsilon(s)), \\ t_\varepsilon(a) = b + \varepsilon. \end{cases} \tag{4.13}$$

We define $\gamma_\varepsilon(s) = (s, t_\varepsilon(s))$ so that $\gamma_0 = \gamma$. Here we exploit an argument similar to the one developed in [65]. By Theorem 2.8 in [82] we gain that t_ε is Lipschitz with respect to ε with Lipschitz constant less than or equal to e^{Lr}. Fix $s \in]a-r, a+r[$, the inverse of the function $\varepsilon \to t_\varepsilon(s)$ is given by $\bar{\chi}_t(-s) = \chi_t(-s) - b$ where χ_t is the unique solution of the following Cauchy problem

$$\begin{cases} \chi_t'(\tau) = 2u(\tau, \chi_t(\tau)) \\ \chi_t(a+s) = t. \end{cases} \tag{4.14}$$

Again by Theorem 2.8 in [82] we have that $\bar{\chi}_t$ is Lipschitz continuous with respect to t, thus the function $\varepsilon \to t_\varepsilon$ is a locally bi-Lipschitz homeomorphisms.

We consider the following Lipschitz coordinates

$$G(\xi,\varepsilon)=(\xi,t_\varepsilon(\xi))=(s,t) \tag{4.15}$$

around the characteristic curve passing through (a,b). Notice that, by the uniqueness result for (4.13), G is injective. Given (s,t) in the image of G using the inverse function $\bar{\chi}_t$ defined in (4.14) we find ε such that $t_\varepsilon(s)=t$, therefore G is surjective. By the Invariance of Domain Theorem [**5**], is a homeomorphism. The Jacobian of G is defined by

$$\mathbf{J}_G=\det\begin{pmatrix} 1 & 0\\ t_\varepsilon' & \frac{\partial t_\varepsilon}{\partial \varepsilon}\end{pmatrix}=\frac{\partial t_\varepsilon}{\partial\varepsilon}(s) \tag{4.16}$$

almost everywhere in ε. Any function φ defined on D can be considered as a function of the variables (ξ,ε) by making $\tilde{\varphi}(\xi,\varepsilon)=\varphi(\xi,t_\varepsilon(\xi))$. Since the function G is C^1 with respect to ξ we have

$$\frac{\partial\tilde{\varphi}}{\partial\xi}=\varphi_x+t_\varepsilon'\,\varphi_t=\varphi_x+2u\varphi_t.$$

Furthermore, by [**28**, Theorem 2 in Section 3.3.3] or [**50**, Theorem 3], we may apply the change of variables formula for Lipschitz maps. Assuming that the support of v is contained in a sufficiently small neighborhood of (a,b), we can express the integral (4.12) as

$$\int_I\Big(\int_{a-r}^{a+r}\big(\big(\frac{\partial\tilde{v}}{\partial\xi}+2\tilde{v}\,\tilde{u}_t\big)\tilde{M}+\tilde{f}\tilde{v}\big)\frac{\partial t_\varepsilon}{\partial\varepsilon}\,d\xi\Big)d\varepsilon=0, \tag{4.17}$$

where I is a small interval containing 0. Instead of $\tilde{v}$ in (4.17) we consider the function $\tilde{v}h/(t_{\varepsilon+h}-t_\varepsilon)$, where h is a small enough parameter. Then we obtain

$$\begin{aligned}\frac{\partial}{\partial\xi}\left(\frac{\tilde{v}h}{(t_{\varepsilon+h}-t_\varepsilon)}\right)&=\frac{\partial\tilde{v}}{\partial\xi}\frac{h}{(t_{\varepsilon+h}-t_\varepsilon)}-\tilde{v}h\frac{t_{\varepsilon+h}'-t_\varepsilon'}{(t_{\varepsilon+h}-t_\varepsilon)^2}\\ &=\frac{\partial\tilde{v}}{\partial\xi}\frac{h}{(t_{\varepsilon+h}-t_\varepsilon)}-2\tilde{v}h\frac{u(\xi,t_{\varepsilon+h}(\xi))-u(\xi,t_\varepsilon(\xi))}{(t_{\varepsilon+h}-t_\varepsilon)^2},\end{aligned}$$

that tends to

$$\left(\frac{\partial t_\varepsilon}{\partial\varepsilon}\right)^{-1}\left(\frac{\partial\tilde{v}}{\partial\xi}-2\tilde{v}\tilde{u}_t\right)\qquad a.e.\text{ in }\varepsilon,$$

when h goes to 0. Putting $\tilde{v}h/(t_{\varepsilon+h}-t_\varepsilon)$ in (4.17) instead of $\tilde{v}$ we gain

$$\int_I\left(\int_{a-r}^{a+r}\frac{h\frac{\partial t_\varepsilon}{\partial\varepsilon}}{(t_{\varepsilon+h}-t_\varepsilon)}\left(\frac{\partial\tilde{v}}{\partial\xi}+2\tilde{v}\,(\tilde{u}_t-\frac{\tilde{u}(\xi,\varepsilon+h)-\tilde{u}(\xi,\varepsilon)}{(t_{\varepsilon+h}-t_\varepsilon)})\right)\tilde{M}+\tilde{f}\tilde{v}\,d\xi\right)d\varepsilon=0.$$

Using Lebesgue's dominated convergence theorem and letting $h\to0$ we have

$$\int_I\left(\int_{a-r}^{a+r}\frac{\partial\tilde{v}}{\partial\xi}\tilde{M}+\tilde{f}\tilde{v}\,d\xi\right)d\varepsilon=0. \tag{4.18}$$

Let $\eta:\mathbb{R}\to\mathbb{R}$ be a positive function compactly supported in I and for $\rho>0$ we consider the family $\eta_\rho(x)=\rho^{-1}\eta(x/\rho)$, that weakly converge to the Dirac delta distribution. Putting the test functions $\eta_\rho(\varepsilon)\psi(\xi)$ in (4.18) and letting $\rho\to0$ we get

$$\int_{a-r}^{a+r}\psi'(\xi)\tilde{M}(\xi,0)+\tilde{f}(\xi,0)\psi(\xi)\,d\xi=0, \tag{4.19}$$

for each $\psi \in C_0^\infty((a-r,a+r))$. Since $u_x + 2uu_t$ is continuous, M in (4.10) is continuous, thus also $\tilde{M}$. Hence thanks to Lemma 4.8 we conclude that M is C^1 along γ, thus by Remark 4.4 is also C^1 along Γ.

Since M is C^1 along the characteristic curve, we can integrate by parts in equation (4.19) to obtain

$$\int_{a-r}^{a+r} \left(-\tilde{M}'(0,\xi) + \tilde{f}(0,\xi)\right) \psi(\xi)\, d\xi = 0,$$

for each $\psi \in C_0^\infty((a-r,a+r))$. That means that M satisfies the equation

$$\frac{d}{ds} M(\gamma(s)) = f(\gamma(s))$$

along characteristic curves. □

Lemma 4.8 ([**42**, Lemma 4.2]). *Let $J \subset \mathbb{R}$ be an open interval and $g, h \in C^0(J)$. Let $H \in C^1(J)$ be a primitive of h. Assume that*

$$\int_J \psi' g + h\psi = 0,$$

for each $\psi \in C_0^\infty(J)$. Then the function $g - H$ is a constant function in J. In particular $g \in C^1(J)$.

Remark 4.9. Let K be a convex body of class C^2_+ such that $0 \in K$. Following [**72**] we consider a clockwise-oriented P-periodic parameterization $\gamma : \mathbb{R} \to \mathbb{R}^2$ of ∂K. For a fixed $v \in \mathbb{R}$ we take the translated curve $s \to \gamma(s+v)-\gamma(v) = (x(s), y(s))$ and we consider its horizontal lifting $\Gamma_v(s)$ to $\mathbb{H}^1$ starting at $(0,0,0) \in \mathbb{H}^1$ for $s = 0$, given by

$$\Gamma_v(s) = \left((x(s), y(s), \int_0^s y(\tau)x'(\tau) - x(\tau)y'(\tau) d\tau\right).$$

The Pansu-Wulff shape associated to K is defined by

$$\mathbb{S}_K = \bigcup_{v \in [0,P)} \Gamma_v([0,P]).$$

In [**72**, Theorem 3.14] it is shown that the horizontal liftings Γ_v, for each $v \in [0,P)$, are solutions for $H_K = 1$, therefore $\mathbb{S}_K$ has constant prescribed K-mean curvature equal to 1. Since the curves Γ_v have the same regularity as ∂K, the C^2 regularity result for horizontal curves obtained in Theorem 4.5 is optimal.

Corollary 4.10. *Let K be a C^2_+ convex set in $\mathbb{R}^2$ with $0 \in \operatorname{int}(K)$ and $||\cdot||_K$ the associated left-invariant norm in $\mathbb{H}^1$. Let $\Omega \subset \mathbb{H}^1$ be an open set and $E \subset \Omega$ a set of prescribed K-mean curvature $f \in C^0(\Omega)$ with C^1 boundary S. Then the characteristic curves in $S \smallsetminus S_0$ are of class C^2.*

Proof. Since S is of class C^1, in the regular part $S \smallsetminus S_0$ the horizontal normal ν_h is a nowhere-vanishing continuous vector fields, thus $S \smallsetminus S_0$ is an $\mathbb{H}$-regular surface. In particular a C^1 surface is Lipschitz, thus $S \smallsetminus S_0$ verifies the hypotheses of Theorem 4.5 and the characteristic curves in $S \smallsetminus S_0$ are of class C^2. □

Remark 4.11. When S is of class C^1 the proof of Lemma 4.7 is is much easier. Indeed the solution t_ε of the Cauchy Problem (4.13) is differentiable in ε, thus the function $\partial t_\varepsilon/\partial\varepsilon$ satisfies the following ODE

$$\left(\frac{\partial t_\varepsilon}{\partial \varepsilon}\right)'(s) = 2u_t(s, t_\varepsilon(s))\frac{\partial t_\varepsilon}{\partial \varepsilon}, \qquad \frac{\partial t_\varepsilon}{\partial \varepsilon}(a) = 1.$$

That implies that

$$\frac{\partial t_\varepsilon}{\partial \varepsilon}(s) = e^{\int_a^s 2u_t(\tau, t_\varepsilon(\tau)))d\tau} > 0.$$

Since the Jacobian $\mathbf{J}_G$ defined in (4.16) is equal to $\partial t_\varepsilon/\partial\varepsilon > 0$ the change of variables $G(\xi, \varepsilon)$ is invertible. Hence the rest of the proof of Lemma 4.7 goes in the same way as before.

4.3. The sub-Finsler mean curvature equation. Given an Euclidean Lipschitz boundary S whose characteristic curves in $S \smallsetminus S_0$ are of class C^2, for each point $p \in S \smallsetminus S_0$ we can define the K-mean curvature H_K of S by

$$H_K = \langle D_Z \pi_K(\nu_h), Z\rangle = \langle \nabla_Z \pi_K(\nu_h), Z\rangle, \tag{4.20}$$

where ν_h is the outer horizontal unit normal to S. This definition was given in [**72**] for surfaces of class C^2.

Proposition 4.12. *Let $\Omega \subset \mathbb{H}^1$ be an open set and $E \subset \Omega$ a set of prescribed K- mean curvature $f \in C^0(\Omega)$ Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then $H_K(p) = f(p)$ for each $p \in S \smallsetminus S_0$.*

Proof. By the Implicit Function Theorem for $\mathbb{H}$-regular surfaces, Theorem 6.5 in [**36**], given a point $p \in S$, after a rotation about the t-axis, there exists an open neighborhood $B \subset \mathbb{H}^1$ of p such that $B \cap S$ is the intrinsic graph of a function $u : D \to \mathbb{R}$ where D is a domain in the vertical plane $y = 0$. The function u is Euclidean Lipschitz by our assumption. We set $B \cap S = \mathrm{Gr}(u)$. We assume that E is locally the epigraph of u.

Let $\Gamma(s)$ be a characteristic curve passing through p in $\mathrm{Gr}(u)$ and $\gamma(s)$ its projection on the xt-plane. The characteristic vector Z defined in (4.7) is given by

$$Z = \frac{X + (u_x + 2uu_t)Y}{(1 + (u_x + 2uu_t)^2)^{\frac{1}{2}}}.$$

Since S is $\mathbb{H}$-regular, Z and the horizontal unit normal

$$\nu_h = \frac{(u_x + 2uu_t)X - Y}{(1 + (u_x + 2uu_t)^2)^{\frac{1}{2}}}$$

are continuous vector fields. By Lemma 4.7 we have that $M = F(u_x + 2uu_t)$ defined in (4.10) satisfies the differential equation

$$\frac{d}{ds} M(\gamma(s)) = f(\gamma(s))$$

along the characteristic curves. Therefore we obtain

$$\begin{aligned}\frac{d}{ds} M(\gamma(s)) &= F'(u_x + 2uu_t)\frac{d}{ds}\big[(u_x + 2uu_y)(\gamma(s))\big] \\ &= \frac{1}{\phi''(u_x + 2uu_t)}\frac{d}{ds}\big[(u_x + 2uu_y)(\gamma(s))\big],\end{aligned}$$

As in proof of Lemma 4.6, we parametrize the lower part of the boundary of the convex body K by a function ϕ defined on a closed interval $I \subset \mathbb{R}$. Again by Lemma 4.6 we have

$$\pi_K(x,-1) = (\varphi(x), \phi(\varphi(x)),$$

where φ is the inverse function of ϕ'. Furthermore the K-mean curvature defined (4.20) is equivalent to

$$\begin{aligned} H_K &= \langle D_Z \pi_K(u_x + 2uu_t, -1), Z\rangle \\ &= \frac{\langle \dfrac{D}{ds}\big[\varphi(u_x+2uu_t)X_\gamma + \phi(\varphi(u_x+2uu_t))Y_\gamma\big], Z\rangle}{1+(u_x+2uu_t)^2} \\ &= \frac{\varphi'(u_x+2uu_t)\dfrac{d}{ds}(u_x+2uu_t)\big(1+\phi'(\varphi(u_x+2uu_t))(u_x+2uu_t)\big)}{1+(u_x+2uu_t)^2} \\ &= \frac{1}{\phi''(u_x+2uu_t)}\frac{d}{ds}\big[(u_x+2uu_t)(\gamma(s))\big]. \end{aligned}$$

Hence we obtain $H_K = \frac{d}{ds}M(\gamma(s))$ and so $H_K(p) = f(p)$ for each $p \in S \smallsetminus S_0$. □

The following result allows us to express the K-mean curvature H_K in terms of the sub-Riemannian mean curvature H_D.

PROPOSITION 4.13. *Let $K \subset \mathbb{R}^2$ be a convex body of class C^2_+ such that $0 \in \text{int}(K)$ and $\pi_K = N_K^{-1}$. Let κ be the strictly positive curvature of the boundary ∂K. Let $\Omega \subset \mathbb{H}^1$ be an open set and $E \subset \Omega$ a set of prescribed K-mean curvature $f \in C^0(\Omega)$ with Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then, we have*

$$H_D(p) = \kappa(\pi_K(\nu_h))f(p) \qquad \textit{for each} \quad p \in S \smallsetminus S_0,$$

where $H_D(p) = \langle D_Z \nu_h, Z\rangle$ is the sub-Riemannian mean curvature, ν_h be the horizontal unit normal at p to $S \smallsetminus S_0$ and $Z = J(\nu_h)$ be the characteristic vector field.

PROOF. By Proposition 4.12 we have $H_K(p) = f(p)$ for each $p \in S \smallsetminus S_0$. We remark that Theorem 4.5 implies that H_K is well-defined.

Let $\gamma : (-\varepsilon, \varepsilon) \to S \smallsetminus S_0$ be the integral curve of Z passing through p, namely $\gamma'(s) = Z_{\gamma(s)}$ and $\gamma(0) = p$. Let $\nu_h(s) = -J(Z_{\gamma(s)})$ be the horizontal unit normal along γ and let

$$\pi(\nu_h(s)) = \pi_1(\nu_h(s))X_{\gamma(s)} + \pi_2(\nu_h(s))Y_{\gamma(s)}.$$

Noticing that $\nabla X = \nabla Y = 0$ we gain

$$\frac{\nabla}{ds}\Big|_{s=0}\pi(\nu_h(s)) = \frac{d}{ds}\Big|_{s=0}\pi_1(\nu_h(s))X_{\gamma(0)} + \frac{d}{ds}\Big|_{s=0}\pi_2(\nu_h(s))Y_{\gamma(0)}.$$

Setting $\nu_h = aX + bY$ we obtain

$$\frac{\nabla}{ds}\Big|_{s=0}\pi(\nu_h(s)) = (d\pi)_{(a,b)}\left(\frac{\nabla}{ds}\Big|_{s=0}\nu_h(s)\right), \tag{4.21}$$

where

$$(d\pi)_{(a,b)} = \begin{pmatrix} \dfrac{\partial \pi_1}{\partial a}(a,b) & \dfrac{\partial \pi_1}{\partial b}(a,b) \\ \dfrac{\partial \pi_2}{\partial a}(a,b) & \dfrac{\partial \pi_2}{\partial b}(a,b) \end{pmatrix}.$$

Moreover, by Corollary 1.7.3 in [78] we get $\pi_K = \nabla h$, where h is a C^2 function. Thus by Schwarz's theorem the Hessian $\text{Hess}_{(a,b)}(h) = (d\pi)_{(a,b)}$ is symmetric, i.e. $(d\pi) = (d\pi)^*$. Equation (4.21) then implies

$$H_K = \langle \nabla_Z \, \pi_K(\nu_h), Z\rangle = \langle \nabla_Z \nu_h, (d\pi)^*_{\nu_h} Z\rangle = \langle \nabla_Z \nu_h, (d\pi)_{\nu_h} Z\rangle.$$

Finally, by Lemma 4.14 we get

$$H_K = \frac{1}{\kappa(\pi_K(\nu_h))}\langle \nabla_Z \nu_h, Z\rangle.$$

Hence we obtain $\langle D_Z \nu_h, Z\rangle = \kappa(\pi_K(\nu_h))$, since $D_Z\nu_h = \nabla_Z \nu_h$. □

LEMMA 4.14. *Let $K \subset \mathbb{R}^2$ be a convex body of class C^2_+ such that $0 \in \text{int}(K)$ and N_K be the Gauss map of ∂K. Let κ be the strictly positive curvature of the boundary ∂K. Let S be an $\mathbb{H}$-regular surface with horizontal unit normal ν_h and characteristic vector field $Z = J(\nu_h)$. Then we have*

$$(d\pi)_{\nu_h} Z = \frac{1}{\kappa} Z \quad \textit{and} \quad (d\pi)_{\nu_h}\nu_h = 0,$$

where $(d\pi)_{\nu_h}$ is the differential of $\pi_K = N_K^{-1}$.

PROOF. Let $\alpha(t) = (x(t), y(t))$ be an arc-length parametrization of ∂K such that $\dot{x}^2(t) + \dot{y}^2(t) = 1$. Let $\nu_h = aX + bY$ be the horizontal unit normal to S, with $a = \cos(\theta)$ and $b = \sin(\theta)$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Notice that $\theta = \arctan(\frac{b}{a})$. Then we have

$$\pi_K(a,b) = N_K^{-1}((a,b)).$$

Let $\varphi : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be the function satisfying

$$\pi_K(\cos(\theta), \sin(\theta)) = (x(\varphi(\theta)), y(\varphi(\theta))).$$

If we consider the normal N_K of the previous equality we obtain

$$(\cos(\theta), \sin(\theta)) = (\dot{y}(\varphi(\theta)), -\dot{x}(\varphi(\theta))).$$

Therefore we have

$$\theta = \arctan\left(-\frac{\dot{x}}{\dot{y}}(\varphi(\theta))\right)$$

for each $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. That means that φ is the inverse of the function $\arctan(-\frac{\dot{x}}{\dot{y}}(t))$, that is invertible since

$$\frac{d}{dt}\arctan(-\tfrac{\dot{x}}{\dot{y}}(t)) = \dot{x}\ddot{y} - \dot{y}\ddot{x} = \kappa(t) > 0.$$

Let $Z = J(\nu_h) = -bX + aY$ be the characteristic vector field, then we have $(d\pi)_{(a,b)} = (d\pi)^*_{(a,b)}$ and

$$(d\pi)^*_{(a,b)} Z = \begin{pmatrix} -b\dfrac{\partial \pi_1}{\partial a} + a\dfrac{\partial \pi_2}{\partial a} \\ -b\dfrac{\partial \pi_1}{\partial b} + a\dfrac{\partial \pi_2}{\partial b} \end{pmatrix},$$

where

$$\pi_1(a,b) = x(\varphi(\arctan(\tfrac{b}{a}))), \qquad \pi_2(a,b) = y(\varphi(\arctan(\tfrac{b}{a}))).$$

Thus we get

$$(d\pi)^*_{(a,b)} Z = \varphi'(\arctan(\tfrac{b}{a}))Z = \frac{1}{\kappa(\varphi(\theta))}Z.$$

A similar straightforward computation shows that $(d\pi)_{\nu_h}\nu_h = 0$. □

5. Cones

5.1. The first variation formula and a stationary condition. In this section we present some consequences of the first variation formula. We assume that the Heisenberg group $\mathbb{H}^1$ is endowed with the sub-Finsler structure associated to a convex set K of class C^2_+ with $0 \in \text{int}(K)$. Recall that, given a surface $S \subset \mathbb{H}^1$ of class C^1, its *singular set* S_0 is composed of those points of S where the tangent plane is horizontal. The *regular* part of S is $S \smallsetminus S_0$.

THEOREM 5.1 (Theorem 3.1 in [**72**]). *Let S be an oriented surface of class C^1 such that the regular part $S \smallsetminus S_0$ is of class C^2. Consider a C^2 vector field U with compact support on S, normal component $u = \langle U, N\rangle$, and associated flow $\{\varphi_s\}_{s\in\mathbb{R}}$. Let $\eta = \pi(\nu_h)$, where ν_h is the horizontal unit normal to S. Then we have*

$$\frac{d}{ds}\bigg|_{s=0} A_K(\varphi_s(S)) = \int_{S\setminus S_0} H_K u \, dS - \int_{S\setminus S_0} \text{div}_S(u\eta^\top)\, dS, \tag{5.1}$$

where div_S is the Riemannian divergence on S and the superscript $\top$ indicates the projection over the tangent plane to S. The quantity $H_K = \langle \nabla_Z \pi(\nu_h), Z\rangle$, for $Z = -J(\nu_h)$, is the K-mean curvature of S.

Using Theorem 5.1 we can prove the following necessary and sufficient condition for a surface S to be A_K-stationary. When a surface S of class C^1 is divided into two parts S^+, S^- by a singular curve S_0 so that S^+, S^- are of class C^2 up to the boundary, the tangent vectors Z^+, Z^- can be chosen so that they parameterize the characteristic curves (i. e., horizontal curves en the regular part of S) as curves leaving from S_0, see Corollary 3.6 in [**12**] . In this case $\eta^+ = \pi(\nu_h) = \pi(J(Z^+))$ and $\eta^- = \pi(J(Z^-))$.

COROLLARY 5.2. *Let S be an oriented surface of class C^1 such that the singular set S_0 is a C^1 curve. Assume that $S \smallsetminus S_0$ is the union of two surfaces S^+, S^- of class C^2 meeting along S_0. Let η^+, η^- the restrictions of η to S^+ and S^-, respectively. Then S is area-stationary if and only if*

(1) *$H_K = 0$, and*
(2) *$\eta^+ - \eta^-$ is tangent to S_0.*

In particular, condition $H_K = 0$ implies that $S \smallsetminus S_0$ is foliated by horizontal straight lines.

PROOF. We may apply the divergence theorem to the second term in (5.1) to get

$$\frac{d}{ds}\bigg|_{s=0} A_K(\varphi_s(S)) = \int_{S\setminus S_0} H_K u \, dS - \int_{S_0} u \, \langle \xi, (\eta^+ - \eta^-)^\top\rangle \, dS,$$

where ξ is the outer unit normal to S^+ along S_0. Hence the stationary condition is equivalent to $H = 0$ on $S \smallsetminus S_0$ and $\langle \xi, \eta^+ - \eta^-\rangle = 0$. The latter condition is equivalent to that $\eta^+ - \eta^-$ be tangent to S_0.

That $H_K = 0$ implies that $S \smallsetminus S_0$ is foliated by horizontal straight lines was proven in Theorem 3.14 in [**72**]. □

Since $\nu^+ = J(Z^+), \nu^- = J(Z^-)$, where Z^+ and Z^- are the extensions of the horizontal tangent vectors in S^+, S^-, we have that the second condition in Corollary 5.2 is equivalent to

$$\pi(J(Z^+)) - \pi(J(Z^-)) \text{ is tangent to } S_0. \tag{5.2}$$

So a natural question is, given a C^2_+ convex body K containing 0 in its interior, and a unit vector $v \in \mathbb{S}^1$, can we find a pair of unit vectors Z^+, Z^- such that (5.2) is satisfied? If such vectors exist, how many pairs can we get? The answer follows from the next result.

Lemma 5.3. *Let K be a convex body of class C^2_+ such that $0 \in \operatorname{int}(K)$. Given $v \in \mathbb{R}^2 \setminus \{0\}$, let $L \subset \mathbb{R}^2$ be the vector line generated by v. Then, for any $u \in \partial K$, we have the following possibilities*

(1) *The only $w \in \partial K$ such that $w - u \in L$ is $w = u$, or*
(2) *There is only one $w \in \partial K$, $w \neq u$ such that $w - u \in L$.*

The first case happens if and only if L is parallel to the support line of K at u.

Proof. Let T be the translation in $\mathbb{R}^2$ of vector u. Then $T(L)$ is a line that meets ∂K at u. The line $T(L)$ intersects ∂K only once when L is the supporting line of $T(K)$ at 0; otherwise L intersects ∂K just at another point $w \neq u$ so that $w - u \in L$. □

Remark 5.4. We use Lemma 5.3 to understand the behavior of characteristic curves meeting at a singular point $p \in S_0$. Let Z^+, Z^- be the tangent vectors to the characteristic lines starting from p. Let ν^+, ν^- be the vectors $J(Z^+), J(Z^-)$, and L the line generated by the tangent vector to S_0 at p. The condition that S is stationary implies that $\eta^+ - \eta^- \in L$. If $w = \eta^+$ and $u = \eta^-$ are equal then $\nu^+ = \nu^-$ are orthogonal to L, which implies that Z^+, Z^- lie in L. This is not possible since characteristic lines meet tranversaly the singular line, again by Corollary 3.6 in [**12**].

Hence $\eta^+ \neq \eta^-$ and η^+ is uniquely determined from η^- by Lemma 5.3. Obviously the roles of η^+ and η^- are interchangeable.

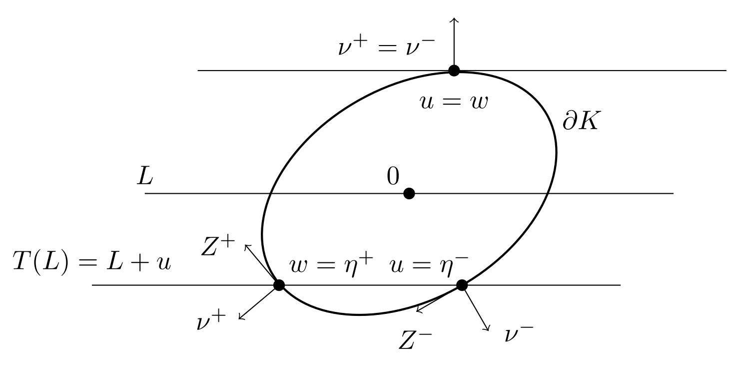


Figure 11. Geometric construction to obtain $w = \eta^+$ from $u = \eta^-$ so that the stationary condition is satisfied. The case $\nu^+ = \nu^-$ cannot hold.

5.2. Examples of entire K-perimeter minimizing horizontal graphs with one singular line. Remark 5.4 implies that Z^- can be uniquely determined from Z^+ when S is a stationary surface. Let us see that this result can be refined to provide a smooth dependence of the oriented angle $\angle(v, Z^-)$ in terms of $\angle(v, Z^+)$. We use complex notation for horizontal vectors assuming that the horizontal distribution is positively oriented by $v, J(v)$ for any $v \in \mathcal{H} \setminus \{0\}$.

LEMMA 5.5. *Let K be a convex body of class C^2_+ with $0 \in \operatorname{int}(K)$. Consider a unit vector $v \in \mathbb{R}^2$ and let $L \subset \mathbb{R}^2$ be the vector line generated by v. Then, for any $\alpha \in (0, \pi)$ there exists a unique $\beta \in (\pi, 2\pi)$ such that if $Z^+ = ve^{i\alpha}$, $Z^- = ve^{i\beta}$, then $\pi(J(Z^+)) - \pi(J(Z^+))$ belongs to L.*

Moreover the function $\beta : (0, \pi) \to (\pi, 2\pi)$ is of class C^1 with negative derivative.

PROOF. We change coordinates so that L is the line $y = 0$. We observe that $Z^+ = ve^{i\alpha}$ implies that $J(Z^+) = ve^{i(\alpha+\pi/2)}$. We define $(x, y) : \mathbb{S}^1 \to \partial K$ by

$$(x(\alpha), y(\alpha)) = N_K^{-1}(ve^{i(\alpha+\pi/2)}),$$

where $N_K : \partial K \to \mathbb{S}^1$ is the (outer) Gauss map of ∂K. The functions x, y are C^1 since N_K is C^1. The point $(x(\alpha), y(\alpha))$ is the only one in ∂K such that the clockwise oriented tangent vector to ∂K makes an angle α with the positive direction of the line L. A line parallel to L meets ∂K at a single point only when $\alpha + \pi/2 = \pi/2$ or $\alpha + \pi/2 = 3\pi/2$. Hence, for $\alpha \in (0, \pi)$, there is a unique $\beta \in (\pi, 2\pi)$ such that

$$(x(\beta), y(\beta)) - (x(\alpha), y(\alpha)) \in L.$$

Observe that, for $\alpha \in (0, \pi)$, we have $dy/d\alpha > 0$ and, for $\beta \in (\pi, 2\pi)$, we get $dy/d\beta < 0$. We can use the implicit function theorem (applied to $y(\beta) - y(\alpha)$) to conclude that β is a C^1 function of α. Moreover

$$\frac{d\beta}{d\alpha} = \frac{dy/d\alpha}{dy/d\beta} < 0,$$

as desired. □

Now we give the main construction in this section.

We fix a vector $v \in \mathbb{R}^2 \setminus \{0\}$ and the line $L_v = \{\lambda v : \lambda \in \mathbb{R}\}$. For every $\lambda \in \mathbb{R}$, we consider two half-lines, $r^+_\lambda, r^-_\lambda \subset \mathbb{R}^2$, extending from the point $p = \lambda v \in L_v$ with angles $\alpha(\lambda)$ and $\beta(\lambda)$ respectively. Here $\alpha : \mathbb{R} \to (0, \pi)$ is a non-increasing function and $\beta(\lambda)$ is the composition of $\alpha(\lambda)$ with the function obtained in Lemma 5.5. Hence $\beta(\lambda)$ is a non-decreasing function. The line L_v can be lifted to the horizontal straight line $R_v = L_v \times \{0\} \subset \mathbb{H}^1$ passing through the point $(0, 0, 0)$, and the half-lines $r^\pm_\lambda$ can be lifted to horizontal half-lines $R^\pm_\lambda$ starting from the point $(\lambda v, 0)$ in the line R_v.

The surface obtained as the union of the half-lines R^+_λ and R^-_λ, for $\lambda \in \mathbb{R}$, is denoted by $\Sigma_{v,\alpha}$. Since any $R^\pm_\lambda$ is a graph over $r^\pm_\lambda$ and $\bigcup_{\lambda\in\mathbb{R}}(r^+_\lambda \cup r^-_\lambda)$ covers the xy-plane, we can write the surface $\Sigma_{v,\alpha}$ as the graph of a continuous function $u_\alpha : \mathbb{R}^2 \to \mathbb{R}$. Writing $v = e^{i\alpha_0}$, the surface $\Sigma_{v,\alpha}$ can be parametrized by $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ as follows

$$\Psi(\lambda, \mu) = \begin{cases} \big(\lambda e^{i\alpha_0} + \mu e^{i(\alpha_0+\alpha(\lambda))}, -\mu\lambda \sin \alpha(\lambda)\big), & \mu \geqslant 0, \\ \big(\lambda e^{i\alpha_0} + |\mu| e^{i(\alpha_0+\beta(\lambda))}, -|\mu|\lambda \sin \beta(\lambda)\big), & \mu \leqslant 0. \end{cases} \tag{5.3}$$

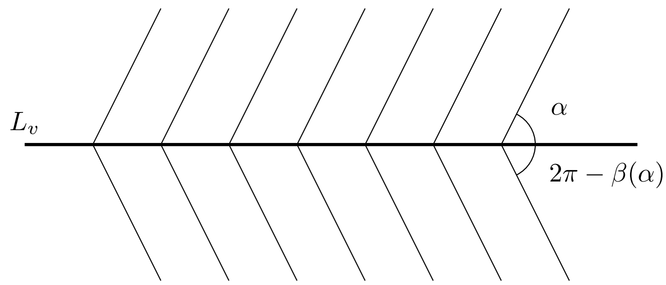


FIGURE 12. The planar configuration to obtain the surface $\Sigma_{v,\alpha}$. Here α is a constant function and K is the unit disk D. Such surfaces were called *herringbone surfaces* by Young [**84**] as they are the union of horizontal rays that branch out of a horizontal line.

EXAMPLE 5.6. A special example to be considered is the sub-Riemannian cone Σ_α, where $\alpha \in (0, \pi)$. The projection of Σ_α to the horizontal plane $t = 0$ is composed of the line $y = 0$ and the half-lines starting from points in $y = 0$ with angles α and $-\alpha$. This cone can be parametrized, for $s \in \mathbb{R}, t \geqslant 0$, by

$$(u, v) \mapsto (u + v\cos\alpha, v\sin\alpha, -uv\sin\alpha)$$

when $y \geqslant 0$, and by

$$(u + v\cos\alpha, -v\sin\alpha, uv\sin\alpha)$$

when $y \leqslant 0$. A straigthforward computation implies that Σ_α is the t-graph of the function

$$u_\alpha(x, y) = -xy + \cot\alpha\, y|y|. \tag{5.4}$$

Observe that

$$\lim_{\alpha\to 0} u_\alpha(x, y) = \begin{cases} +\infty, & y > 0, \\ 0, & y = 0, \\ -\infty, & y < 0, \end{cases} \tag{5.5}$$

so that the subgraph of Σ_α converges pointwise locally when $\alpha \to 0$ to a vertical half-space.

The following rsult provides some properties of u_α when $\alpha(\lambda)$ is a smooth function of λ.

PROPOSITION 5.7. *Let $\alpha \in C^k(\mathbb{R})$, $k \geqslant 2$, be a non-decreasing function. Then*
i) u_α is a C^k function in $\mathbb{R}^2 \setminus L_v$,
ii) u_α is merely $C^{1,1}$ near L_v when $\beta \neq \alpha + \pi$.
iii) u_α is C^∞ in any open set I of values of λ when $\beta = \alpha + \pi$ on I.
iv) $\Sigma_{v,\alpha}$ is K-perimeter-minimizing when $\beta = \beta(\alpha)$.
v) The projection of the singular set of $\Sigma_{v,\alpha}$ to the xy-plane is L_v.

PROOF. i), ii), iii) and v) are proven in Lemma 3.1 in [**73**].

We prove iv) by a calibration argument. We shall drop the subscript α to simplify the notation. Let E be the subgraph of u and $F \subseteq \mathbb{H}^1$ such that $F = E$ outside a Euclidean ball centered at the origin. Let $P = \{(z, t) : \langle z, v\rangle = 0\}$, $P^1 = \{(z, t) : \langle z, v\rangle > 0\}$ and $P^2 = \{(z, t) : \langle z, v\rangle < 0\}$. We define two vector fields

U^1, U^2 on P^1, P^2 respectively by vertical translations of the vectors $\pi(\nu_E)|_{P^1} = \eta^+$ and $\pi(\nu_E)|_{P^2} = \eta^-$. They are C^2 in the interior of the half-spaces and extend continuously to the boundary plane P. As $\operatorname{div}(U^j)_{(z,t)}$ coincides with the sub-Finsler mean curvature of the translation of $\Sigma_{v,\alpha}$ passing through (z,t) as defined in [**72**], and these surfaces are foliated by horizontal straight lines in the interior of the half-spaces, by Theorem 3.14 in [**72**] we get

$$\operatorname{div} U^j = 0 \quad j = 1, 2.$$

Here $\operatorname{div} U$ is the Riemannian divergence of the vector field U. We apply the divergence theorem (Theorem 2.1 in [**73**]) to get

$$0 = \int_{F\cap P^j\cap B} \operatorname{div} U^j = \int_F \langle U^j, \nu_{P^j\cap B}\rangle |\partial(P^j \cap B)| + \int_{P^j\cap B} \langle U^j, \nu_F\rangle |\partial F|.$$

Let $C = P \cap \bar{B}$. Then, for every $p \in C$, we have $\nu_{P^1\cap B} = J(v)$ is a normal vector to the plane P and $\nu_{P^2\cap B} = -J(v)$, $U^1 = \eta^+$ and $U^2 = \eta^-$. Hence, by Lemma 5.5, we get

$$\langle U^1, \nu_{P^1\cap B}\rangle + \langle U^2, \nu_{P^2\cap B}\rangle = \langle \eta^+ - \eta^-, J(v)\rangle = 0 \quad p \in C.$$

Adding the above integrals we obtain

$$0 = \sum_{j=1,2} \int_F \langle U^j, \nu_B\rangle d|\partial B| + \int_{B\cap \operatorname{int}(H^j)} \langle U^j, \nu_F\rangle d|\partial F|. \tag{5.6}$$

From the Cauchy-Schwarz inequality and the fact that $|\partial F|$ is a positive measure, we get that

$$\sum_{j=1,2} \int_{B\cap P^j} \langle U^j, \nu_F\rangle d|\partial F| \leqslant P_K(F, B). \tag{5.7}$$

In particular, if we apply the same reasoning to E, equality holds and

$$0 = \sum_{j=1,2} \int_E \langle U^j, \nu_B\rangle d|\partial B| + P_K(E, B). \tag{5.8}$$

From (5.6), (5.7), (5.8) and the fact that $F = E$ in the boundary of B, we get

$$P_K(E, B) \leqslant P_K(F, B),$$

as desired. □

The general properties of $\Sigma_{v,\alpha}$ when α is only continuous are given in the following result.

Proposition 5.8. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous and non-decreasing function. Then*

i) u_α is locally Lipschitz in Euclidean sense,
ii) E_α is a set of locally finite perimeter in $\mathbb{H}^1$, and
iii) $\Sigma_{v,\alpha}$ is K-perimeter-minimizing in $\mathbb{H}^1$.

Proof. i) and ii) are proven in [**73**], Proposition 3.2. Let

$$\alpha_\varepsilon(x) = \int_{\mathbb{R}} \alpha(y)\delta_\varepsilon(x - y)dy$$

the usual convolution, where δ is a Dirac function and $\delta_\varepsilon = \frac{\delta(x/\varepsilon)}{\varepsilon}$. Then α_ε is a C^∞ non-decreasing function and α_ε converges uniformly to α on compact sets of $\mathbb{R}$. By Lemma 5.5, $\beta_\varepsilon = \beta(\alpha_\varepsilon)$ is a C^1 non-decreasing function. Since β is C^1 with respect to α it follows the uniform convergence on compact sets of β_ε to a function $\bar{\beta}$.

Take $F \subset \mathbb{H}^1$ so that $F = E$ outside a Euclidean ball centered at the origin. We follow the arguments of the proof of iv) in Proposition 5.7 and define vector fields $\operatorname{div}(U_\varepsilon^j)$ translating vertically $\pi(\nu_{E_\varepsilon})$, where E_ε is the subgraph of $\Sigma_{\alpha_\varepsilon}$, to obtain by the divergence theorem

$$\sum_{j=1,2} \int_{B\cap \operatorname{int}(P^i)} \langle U_\varepsilon^j, \nu_{E_\varepsilon}\rangle |\partial E_\varepsilon| = \sum_{j=1,2} \int_{B\cap \operatorname{int}(P^i)} \langle U_\varepsilon^j, \nu_F\rangle |\partial F|,$$

the left hand side is the K-perimeter of E_ε, while the right hand side is trivially bounded by the K-perimeter of F. Therefore

$$P_K(E_\varepsilon, B) \leqslant P_K(F, B).$$

Since E_ε converges uniformly in compact sets to E, we obtain the result. □

We study now with some detail the case when $\Sigma_{v,\alpha}$ is a C^∞ surface.

Corollary 5.9. *When α is constant, the surface $\Sigma_{v,\alpha}$ is a K-perimeter-minimizing cone in $\mathbb{H}^1$ of class $C^{1,1}$. The singular set is a horizontal straight line and the regular part of $\Sigma_{v,\alpha}$ is a C^∞ surface.*

The following extends the already known result that in the sub-Riemannian setting the surfaces $\Sigma_{v,\pi/2}$ are C^∞.

Lemma 5.10. *Let $v \in \mathbb{R}^2 \setminus \{0\}$ and $\alpha \in (0,\pi)$ be fixed. If K is centrally symmetric with respect to $O = \frac{1}{2}\eta^+ + \frac{1}{2}\eta^-$ then $\beta(\alpha) = \alpha + \pi$, where $\eta^+ = \pi(J(ve^{i\alpha}))$ and $\eta^- = \pi(J(ve^{i\beta}))$.*

Proof. Let K be centrally symmetric with respect to O. Then η^- is the symmetric point of η^+. On the other hand, the convex body $K - O$ is symmetric with respect to the origin. Then the dual norm is even and, in particular, $\pi_{K-O}(-\nu^+) = -\pi_{K-O}(\nu^+)$. Now, since a translation takes symmetric points of $K - O$ with respect to the origin to symmetric points of K with respect to O, we get $\nu^- = -\nu^+$. This implies that $\beta(\alpha) = \alpha + \pi$. □

The existence of a convex body K of class C^2_+ such that $0 \in \operatorname{int}(K)$ for which $\Sigma_{v,\alpha}$ is C^∞ is studied in Corollary 5.11 and Proposition 5.12.

Corollary 5.11. *Let $v \in \mathbb{R}^2 \setminus \{0\}$ and $\alpha \in (0,\pi)$ be fixed. Then there exists a convex body K of class C^2_+ with $0 \in \operatorname{int}(K)$ such that $\Sigma_{v,\alpha}$ is C^∞.*

Proof. To construct the convex body K, fix a point $p \in \{(x,y) : \langle (x,y), ve^{i\alpha}\rangle > 0\}$ and $O \in J(L) + p \cap L$, where L is the vector line generated by v. Then any K of class C^2_+ centrally symmetric with respect to O containing the origin such that $p \in \partial K$ and $ve^{i\alpha} \perp T_p\partial K$ satisfies the hypothesis of Lemma 5.10, where $\eta^+ = p$ and η^- is the symmetric of η^+ with respect to O. Thus, by (iii) in Proposition 5.7 we get that $\Sigma_{v,\alpha}$ is C^∞. □

Proposition 5.12. *Given a convex body K of class C^2_+ with $0 \in \operatorname{int}(K)$, there exists $v \in \mathbb{R}^2$ such that $\Sigma_{v,\pi/2}$ is C^∞.*

Proof. Let p and q be points in K at maximal distance. Then the lines through p and q orthogonal to $q - p$ are support lines to K. Taking $v = q - p$ and setting $p = \eta^+$ we have $q = \eta^-$, while the vectors ν^+ and ν^- are over the line $L(v)$, that is, Z^+ Z^- make angles $\pi/2$ and $3\pi/2$ with $L(v)$. □

For fixed $v \in \mathbb{R}^2$, we define the surface $\Sigma^+_{v,\alpha}$ as the one composed of all the horizontal half-lines R^+_λ and $R^-_\lambda \subseteq \mathbb{R}^2$ extending from the lifting of the point $p = \lambda v \in L_v$, $\lambda \geqslant 0$, to $\mathbb{H}^1$. The surface $\Sigma^+_{v,\alpha}$ has a boundary composed of two horizontal lines and its singular set is the ray $L^+_v = \{\lambda v : \lambda > 0\}$. We present some pictures of such surfaces.

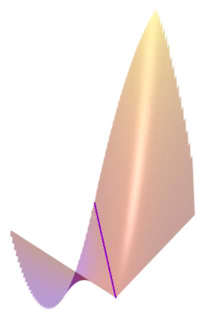


FIGURE 13. The surface $\Sigma^+_{\pi/3,\pi/6}$ associated to the norm $||\cdot||_D$, where D is the unit disk. The singular set corresponds to the purple ray of angle $e^{i\pi/3}$.

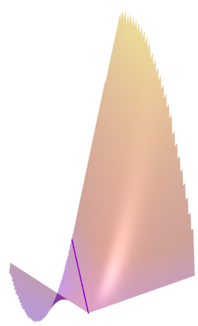


FIGURE 14. The surface $\Sigma^+_{\pi/3,\pi/6}$ associated to the p-norm with $p = 1.5$. The left part of the figure coincides with the left part of Figure 13, while the angle β is bigger. Notice that also the height has increased.

5.3. Area-Minimizing Cones in $\mathbb{H}^1$. We proceed now to construct examples of K-perimeter minimizing cones in $\mathbb{H}^1$ with an arbitrary finite number of horizontal half-lines meeting at the origin. The building blocks for this construction are liftings of circular sectors of the cones considered in Corollary 5.9.

We first prove the following result.

LEMMA 5.13. *Let K be a convex body of class C^2_+ such that $0 \in \text{int}(K)$. Let $u, w \in \mathbb{S}^1$, $\theta = \angle(u,w) > 0$. Then there exists $v \in \mathbb{S}^1$ such that the vector line L_v generated by v splits the sector determined by u and w into two sectors of oriented angles α and β such that $\alpha + \beta = \theta$. Moreover, the stationary condition $\pi(J(u)) - \pi_K(J(w)) \in L_v$ is satisfied.*

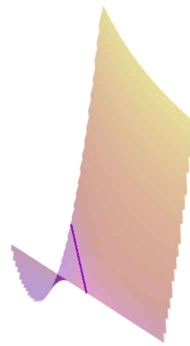


FIGURE 15. The surface $\Sigma^+_{\pi/3,\pi/6}$ with $\beta = \alpha + \pi$. There existence of K is granted by Corollary 5.11.

PROOF. Let $\nu_u = J(u)$, $\nu_w = J(w)$ and $\eta_u = \pi(\nu_u)$, $\eta_w = \pi(\nu_w)$, $\eta_u \neq \eta_w$ since π is a C^1 diffeomorphism. Thus there exists a unique line $\tilde{L}$ passing through η_u and η_w and $L = \tilde{L} - \eta_u$ is a straight line passing though the origin. Notice that $\tilde{L}$ splits ∂K in two connect open components ∂K_1 and ∂K_2. There exist two points $\eta_1 \in \partial K_1$ and $\eta_2 \in \partial K_2$ such that $L + \eta_1$ (resp. $L + \eta_2$) is the support line at η_1 (resp. η_2). Setting $v_1 = N_{\partial K}(\eta_1)$ and $v_2 = N_{\partial K}(\eta_2)$ we gain that v_i for $i = 1, 2$ is perpendicular to L. Without loss of generality we set that $-J(v_1)$ belongs to the portion of plane identified by the θ and $-J(v_2)$ belongs to the portion of plane identified by the $2\pi - \theta$. Then we set $v = -J(v_1)$. Notice that v splits θ in two angles $\beta = \angle(u, v)$, $\alpha = \angle(v, w)$ with $\theta = \alpha + \beta$ and $L = L_v$. □

Now we proceed with the construction inspired by the sub-Riemannian construction in [**49**]. For $k \geqslant 3$ consider a fixed angle θ_0 and family of positive oriented angles $\theta_1, \dots, \theta_k$ such that $\theta_1 + \cdots + \theta_k = 2\pi$. Consider the planar vectors $u_0 = (\cos(\theta_0), \sin(\theta_0))$ and

$$u_i = (\cos(\theta_0 + \theta_1 + \cdots + \theta_i), \sin(\theta_0 + \theta_1 + \cdots + \theta_i)), \qquad i = 1, \dots, k.$$

Observe that $u_k = u_0$. For every $i \in \{1, \dots, k\}$ consider the vectors u_{i-1}, u_i and apply Lemma 5.13 to obtain a family of k vectors v_i in $\mathbb{S}^1$ between u_{i-1} and u_i. We lift the half-lines $L_i = \{\lambda v_i : \lambda \geqslant 0\}$ to horizontal straight lines passing through $(0, 0, 0) \in \mathbb{H}^1$, and we also lift the half-lines

$$\lambda v_i + \{\rho u_{i-1} : \rho \geqslant 0\}, \qquad \lambda v_i + \{\rho u_i : \rho \geqslant 0\},$$

to horizontal straight lines starting from $(\lambda v_i, 0)$. This way we obtain a surface

$$C_K(\theta_0, \theta_1, \dots, \theta_k)$$

with the following properties

THEOREM 5.14. *The surface $C_K(\theta_0, \theta_1, \dots, \theta_k)$ is K-perimeter-minimizing cone which is the graph of a C^1 function.*

PROOF. $C_K(\theta_0, \theta_1, \dots, \theta_k)$ is a cone by construction. It is an entire graph since it is composed of horizontal lifting of straight half-lines in the xy-plane that covered the whole plane without intersecting themselves transversally. The K-perimeter-minimizing property follows in a similar way to from Proposition 2.4 in [**49**]. That it is the graph of a C^1 function is proven like in Proposition 3.2(4) in [**49**]. □

A particular example of area-minimizing cones are those who uses the sub-Riemannian cones C_α restricted to the circular sector with $\theta \in (-\alpha, \alpha)$ as as model piece of the cone. Taking $K = D$, $k \geqslant 3$, and the angle $\alpha = \pi/k$, we define

$$C(k) = C_D\big(\frac{\pi}{k}, \frac{2\pi}{k}, \dots, \frac{2\pi}{k}\big).$$

Let us denote by u_k the functions in $\mathbb{R}^2$ whose graph is $C(k)$. The behavior when k tends to infinity of u_k in a disk is analyzed in the following result.

Proposition 5.15. *The sequence u_k converge to 0 uniformly on compact subsets of $\mathbb{R}^2$. Moreover, the sub-Riemannian area of u_k converges locally to the sub-Riemannian area of the plane $t = 0$. Moreover the sub-Riemannian area of u_k converges to the one of the plane $t = 0$.*

Proof. Since u_k is obtained by collating some rotated copies of u_α, where $\alpha = \pi/k$, we can estimate the height of u_k by the height of u_α. By (5.4), using polar coordinates (r, θ), where $\theta \in [-\alpha, \alpha]$ and $r < r_0$, we get

$$|u_\alpha| \leqslant 2r_0^2 |\sin(\pi/k)|$$

on $D(r_0) = \overline{B}(0, r_0)$. The claim follows since $\lim_{k\to\infty} \sin(\pi/k) = 0$.

The sub-Riemannian area of the graph of u_k over $D(r_0)$ is given by

$$A_D(u_k, r_0) = \int_{D(r_0)} \|\nabla u_k + (-y, x)\| dxdy.$$

Since the sub-Riemannian perimeter is rotationally invariant, we can decompose the above integral as k times the area of the cone C_α in the circular sector with $\theta \in (-\alpha, \alpha)$ and $r < r_0$. By (5.4), it is immediate that

$$\|\nabla u_k(x, y) + (-y, x)\| = 2|y| \sin^{-1}(\alpha).$$

A direct computation shows that

$$A_D(u_k, r_0) = \frac{4\pi r_0^3}{3} \frac{1 - \cos \pi/k}{(\pi/k) \sin \pi/k}.$$

Then $A_D(u_k, r_0)$ tends to $\frac{2\pi r_0^3}{3}$ as $k \to +\infty$. □

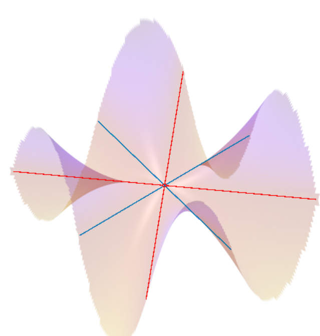


Figure 16. The cone $C(4)$. The singular set is composed of the red rays of angle $0, \pi/2, \pi, (3\pi)/2$, while the rays of angles $\pi/4, (3\pi)/4, (5\pi)/4), (7\pi)/4$, where two pieces of the construction meet, are depicted in cyan.

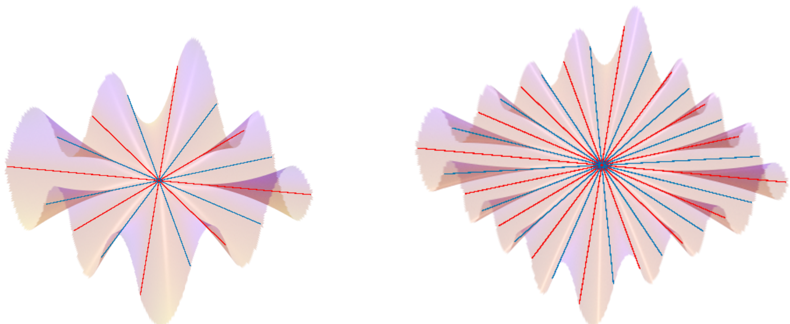


FIGURE 17. The cones $C(8)$ and $C(16)$. They are depicted at the same in this Figure and the previous one. As the number of angles increases, the cone produces more oscilations of smaller height.

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